Application of a Symmetry-Based Method to Stability Analysis of Nonparallel Flows

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Abstract: -
In the symmetry approach to the separations of variables in linear PDEs, a possibility of variable separation in a given PDE is intimately related to its symmetry properties. Namely, solutions with separable variables are common eigenfunctions of $n-1$ mutually commuting linear second order symmetry operators of the equation, and separation constants are eigenvalues of these operators. This method starts with a set of commuting symmetry operators of the equation and finishes by constructing the coordinate systems. In the so-called direct approach to separation of variables in linear PDEs, which formalizes the main features of the symmetry-based approach, a form of the 'ansätz' for a solution with separated variables as well as a form of reduced ODEs, that should be obtained as a result of the variable separation, are postulated from the beginning. Upon obtaining the solutions, they can be related to a set of mutually commuting symmetry operators of the equations under consideration. In the present paper, the 'direct' approach to separation of variables in linear PDEs is applied to the hydrodynamic stability problem. The results of application of the method are the new coordinate systems and the most general forms of basic flows, which permit the postulated form of separation of variables. Then the ODE eigenvalue problems are solved numerically with the help of the spectral collocation method based on Chebyshev polynomials.Calculations are made for the linear stability equations written in Cartesian and cylindrical coordinates.

Key-Words: - hydrodynamic stability, symmetry methods, separation of variables, nonparallel flows

1 Introduction

The linear stability theory (see, e.g., [1]) for a particular flow starts with a solution of the equations of motion representing this basic flow. One then considers this solution with a small perturbation superimposed. Substituting the perturbed solution into the equations of motion and neglecting all terms that involve the square of the perturbation amplitude yield the linear stability equations which govern the behavior of the perturbation. If the perturbation dies away the original flow is said to be stable, and if the perturbation grows, the flow is said to be unstable. Whether a small disturbance that is superimposed upon a known primary flow will be amplified or damped depends on the pattern of the primary flow and the nature of the disturbance. The linearization provides a means of allowing for the many different forms that the disturbance can take. In the method of normal modes, small disturbances are resolved into modes, which may be treated separately because each satisfies the linear equations and there are no interactions between different modes.

Thus, the mathematical problem of the determination of stability of a given flow involves deriving a set of perturbation equations obtained from the Navier-Stokes equations by linearization around this basic flow and finding a set of possible solutions which would permit splitting a perturbation into normal modes. For a steady-state basic flow, normal modes depending on time exponentially, with a complex exponent λ, are permissable - the sign of the real part of λ indicates whether the disturbance grows or decays in time. If further separation of variables is possible, it makes the stability problem amenable to the normal mode analysis in its common form when the problem reduces to that of solving a set of ordinary differential equations. It can be done, however, only for basic flows of specific forms - mostly those are the parallel flows or their axial symmetric counterparts.

For nonparallel basic flows, when the coefficients in the equations for disturbance flow are dependent not only on the normal to the flow coordinate but also on the other coordinates, the corresponding operator does not separate unless certain terms are ignored. If, in addition, the basic flow is non-steady, this brings about great difficulties in theoretical studies of the instability since the normal modes containing an exponential time factor $exp(\lambda t)$ are not applicable here. Therefore stability of viscous incompressible flows developing both in space and time is a little studied topic in the theory of hydrodynamic stability.

In the 1960-1970s a symmetry approach to the separations of variables in linear PDEs has been developed (see, e.g., [2] and references therein). In this method, a possibility of variable separation in a given PDE is intimately related to its symmetry properties. Namely, solutions with separable variables are common eigenfunctions of $n-1$ mutually commuting linear second order symmetry operators of the equation, and separation constants are eigenvalues of these operators. This method starts with a set of commuting symmetry operators of the equation and finishes by constructing the coordinate systems.

Recently, the so-called direct approach to separation of variables in linear PDEs has been developed by a proper formalizing the features of the notion of separation of variables (see, e.g., [3, 4]). In this approach, a form of the 'ansätz' for a solution with separated variables as well as a form of reduced ODEs, that should be obtained as a result of the variable separation, are postulated from the beginning. (Upon obtaining the solutions, they can be related to a set of mutually commuting symmetry operators of the equations under consideration, but a computation of symmetry operators becomes an extra step which is not, in fact, necessary for obtaining solution with separated variables.) The method has been successfully applied to several equations of mathematical physics.

In the present paper, we apply the direct approach [3, 4] to separate the variables in the linear stability equations which govern the disturbance behavior in viscous incompressible fluid flows. The calculations are made both for the Cartesian and cylindrical coordinates. Further we discuss the fluid dynamics interpretation and stability properties of some classes of the exact solutions of the Navier-Stokes equations defined as flows for which the stability analysis can be reduced via separation of variables to the eigenvalue problems of ordinary differential equations. The eigenvalue problems were solved numerically with the help of the spectral collocation method based on Chebyshev polynomials. In some cases, the eigenvalue problems can be solved analytically. Those unique examples of exact (even explicit) solution of the nonparallel unsteady flow stability problems provide a very useful test for numerical methods of solution of eigenvalue problems, and for methods used in the hydrodynamic stability theory, in general.

2 Application of the direct approach to separation of variables in the linear stability equations

2.1 Cartesian coordinates

Throughout the paper we deal with the Navier-Stokes equations governing flows of incompressible Newtonian fluids. Then the equation of motion and the equation of continuity are

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} + (\hat{\mathbf{v}}\nabla)\,\hat{\mathbf{v}} = -\frac{1}{\rho}\nabla\hat{p} + \nu\nabla^2\hat{\mathbf{v}} \text{ and } \nabla\hat{\mathbf{v}} = 0, \quad (1)$$

where ρ is the constant density and ν is the constant kinematic viscosity of the fluid.

As usual in stability analysis, we split the velocity and pressure fields $(\hat{v}_x, \hat{v}_y, \hat{v}_z, \hat{p})$ into two problems: the basic flow problem (V_x, V_y, V_z, P) and a perturbation one (v_x, v_y, v_z, p),

$$\hat{v}_x = V_x + v_x,\ \hat{v}_y = V_y + v_y,\ \hat{v}_z = V_z + v_z,\ \hat{p} = P + p \quad (2)$$

Introducing (2) into the Navier-Stokes equations (1) and neglecting all terms that involve the square of the perturbation amplitude, while imposing the requirement that the basic flow variables (V_x, V_y, V_z, P) themselves satisfy the Navier-Stokes equations, one arrives at the following set of linear stability equations in the Cartesian coordinates:

$$\begin{aligned}
&\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + v_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + v_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial v_x}{\partial z} + \\
&\quad v_z\frac{\partial V_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right), \\
&\frac{\partial v_y}{\partial t} + V_x\frac{\partial v_y}{\partial x} + v_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial v_y}{\partial y} + v_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial v_y}{\partial z} + \\
&\quad v_z\frac{\partial V_y}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right), \\
&\frac{\partial v_z}{\partial t} + V_x\frac{\partial v_z}{\partial x} + v_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial v_z}{\partial y} + v_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial v_z}{\partial z} + \\
&\quad v_z\frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right), \\
&\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \qquad (3)
\end{aligned}$$

Let us introduce a new coordinate system t, $\xi = \xi(t, x), \eta = \eta(t, y), \gamma = \gamma(t, z)$

We choose the Ansatz for a solution $\mathbf{u} = (v_x, v_y, v_z)$ and p to be found

$$\begin{aligned}
\mathbf{u} &= T(t)\exp(a\xi + s\gamma + mS(t))\mathbf{f}(\eta), \\
p &= T_1(t)\exp(a\xi + s\gamma + mS(t))k(\eta) \qquad (4)
\end{aligned}$$

where $\mathbf{f} = (h, f, g)$ and functions $T(t)$, $T_1(t)$, $S(t)$, $\xi(t,x)$, $\eta(t,y)$, $\gamma(t,z)$ are not fixed *a priori* but chosen in such a way that inserting the expressions (4) into system of PDEs (3) yields a system of three second-order and one first order ordinary differential equations for four functions $h(\eta)$, $f(\eta)$, $g(\eta)$, $k(\eta)$. To get constraints on functions T, T_1, S, ξ, η, γ we formalize a reduction procedure as follows.

First, we postulate the form of the resulting system of ordinary differential equations as follows

$$\begin{aligned} h''(\eta) &= U_{11}g'(\eta) + U_{12}h'(\eta) + U_{13}k'(\eta) + \\ &\quad U_{14}f(\eta) + U_{15}g(\eta) + U_{16}h(\eta) + U_{17}k(\eta), \\ f''(\eta) &= U_{21}g'(\eta) + U_{22}h'(\eta) + U_{23}k'(\eta) + \\ &\quad U_{24}f(\eta) + U_{25}g(\eta) + U_{26}h(\eta) + U_{27}k(\eta), \\ g''(\eta) &= U_{31}g'(\eta) + U_{32}h'(\eta) + U_{33}k'(\eta) + \\ &\quad + U_{34}f(\eta)U_{35}g(\eta) + U_{36}h(\eta) + U_{37}k(\eta), \\ f'(\eta) &= U_{41}f(\eta) + U_{42}g(\eta) + U_{43}h(\eta) + U_{44}k(\eta). \end{aligned} \tag{5}$$

Here U_{ij} are second order polynomials with respect to spectral parameters a, s, m with coefficients, which are some smooth functions on η and should be determined on the next steps of the algorithm. Next, we insert the expressions (4) into (3) which yields a system of PDEs containing the functions ξ, η, γ and their first- and second-order partial derivatives, and the functions $f(\eta)$, $g(\eta)$, $k(\eta)$ and their derivatives. Further we replace the derivatives $h''(\eta)$, $f''(\eta)$, $g''(\eta)$, $f'(\eta)$ by the corresponding expressions from the right-hand sides of (5).

Now we regard $h'(\eta)$, $g'(\eta)$, $k'(\eta)$, $h(\eta)$, $f(\eta)$, $g(\eta)$, $k(\eta)$ as the new independent variables. As the functions $\xi(x,t)$, $\eta(y,t)$, $\gamma(z,t)$, $T(t)$, $T_1(t)$, $S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij} (which are functions of η) are independent on these variables, we can require that the obtained equality is transformed into identity under arbitrary $h'(\eta)$, $g'(\eta)$, $k'(\eta)$, $h(\eta)$, $f(\eta)$, $g(\eta)$, $k(\eta)$. In other words, we should split the equality with respect to these variables. After splitting we get an overdetermined system of nonlinear partial differential equations for unknown functions $\xi(x,t)$, $\eta(y,t)$, $\gamma(z,t)$, $T(t)$, $T_1(t)$, $S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij}. At the last step we solve the above system to get an exhaustive description of coordinate systems providing separability of equations (3), as well as all possible basic flows V_x, V_y, V_z such that the system (3) is solvable by the method of separation of variables.

Thus, the problem of variable separation in equation (3) reduces to integrating the overdetermined system of PDEs for unknown functions $\xi(x,t)$, $\eta(y,t)$, $\gamma(z,t)$, $T(t)$, $T_1(t)$, $S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij}. This have been done with the aid of *Mathematica* package.

Below we list the results for the Cartesian coordinates.

The most general form of the basic flow is:

$$\begin{aligned} V_x &= \nu A(\eta)T(t) - \frac{c_1'(t) + xT'(t)}{T(t)}, \\ V_y &= \nu B(\eta)T(t) - \frac{c_2'(t) + yT'(t)}{T(t)}, \\ V_z &= \nu C(\eta)T(t) - \frac{c_3'(t) + zT'(t)}{T(t)}, \end{aligned}$$

The forms of the perturbations v_x, v_y, v_z and p are:

$$\begin{aligned} \mathbf{u} &= T(t)\exp\left(a\xi + s\gamma + m\int T(t)^2 dt\right)\mathbf{f}(\eta), \\ p &= \rho\, T(t)^2 \exp\left(a\xi + s\gamma + m\int T(t)^2 dt\right)k(\eta), \end{aligned}$$

where $\xi = T(t)x + c_1(t)$, $\eta = T(t)y + c_2(t)$, $\gamma = T(t)z + c_3(t)$.

The equations with separated variables are

$$\begin{aligned} &(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta))h(\eta) + \\ &\quad ak(\eta) + \nu(f(\eta)A'(\eta) + B(\eta)h'(\eta) - h''(\eta)) = 0, \\ &f(\eta)(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta) + \\ &\quad \nu B'(\eta)) + \nu B(\eta)f'(\eta) + k'(\eta) - \nu f''(\eta) = 0, \\ &(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta))g(\eta) + \\ &\quad sk(\eta) + \nu(f(\eta)C'(\eta) + B(\eta)g'(\eta) - g''(\eta)) = 0, \\ &sg(\eta) + ah(\eta) + f'(\eta) = 0. \end{aligned} \tag{6}$$

The restrictions on the forms of the basic flows following from the requirement that they themselves satisfy the Navier-Stokes equations lead to the two following cases:

Case I:

$$\xi = \frac{1}{\sqrt{t}}x + c_1(t);\ \eta = \frac{1}{\sqrt{t}}y + c_2(t);\ \gamma = \frac{1}{\sqrt{t}}z + c_3(t).$$

$$\begin{aligned} V_x &= \frac{x}{2t} + \nu A(\eta)\frac{1}{\sqrt{t}} - c_1'(t)\sqrt{t}, \\ V_y &= -\frac{y}{t} - \frac{1}{\sqrt{t}}\left(tc_2'(t) + \frac{3}{2}c_2(t)\right), \\ V_z &= \frac{z}{2t} + \nu C(\eta)\frac{1}{\sqrt{t}} - c_3'(t)\sqrt{t}, \end{aligned} \tag{7}$$

and the functions $A(x)$ and $C(x)$ satisfy the equations

$$3A'(\eta) + 3\eta A''(\eta) + 2\nu A'''(\eta) = 0, \qquad (8)$$

$$3C'(\eta) + 3\eta C''(\eta) + 2\nu C'''(\eta) = 0. \qquad (9)$$

which can be solved in terms of the error functions and the generalized hypergeometric functions. The separation Ansatz takes the form

$$\mathbf{u} = t^s e^{a\xi + m\gamma}\mathbf{f}(\eta),\ p = \rho t^{s-1/2} e^{a\xi+m\gamma}\pi(\eta). \quad (10)$$

For the *Case II* we have $\xi = x + c_1(t);\ \eta = y + c_2(t);\ \gamma = z + c_3(t);\ V_x = A_1\eta^2 + A_2\eta - c_1'(t)$, $V_y = -c_2'(t)$, $V_z = C_1\eta^2 + C_2\eta - c_3'(t)$ and the separation Ansatz is

$$\mathbf{u} = e^{a\xi + s\gamma + mt}\mathbf{f}(\eta),\ p = \rho e^{a\xi + s\gamma + mt}\pi(\eta).$$

2.2 Cylindrical coordinates

The Navier-Stokes equations are written in cylindrical coordinates (r, φ, z) and then the velocity and pressure fields $\hat{v}_r, \hat{v}_\varphi, \hat{v}_z, \hat{p}$ are splitted into the basic flow and perturbation parts

$$\hat{v}_r = V_r + v_r,\ \hat{v}_\varphi = V_\varphi + v_\varphi,\ \hat{v}_z = V_z + v_z,\ \hat{p} = P + p \quad (11)$$

where V_r, V_φ, V_z, P are the basic flow fields and v_r, v_φ, v_z, p are the perturbations. The algorithm for separation of variables in the non-stationary cylindrical coordinate system are similar to those described in the previous section for the Cartesian coordinate system. Below we show the results.

The most general form of the basic flow is:

$$V_z = A(\xi)T(t) - \frac{c'(t) + zT'(t)}{T(t)},$$
$$V_r = B(\xi)T(t) - r\frac{T'(t)}{T(t)},\ V_\varphi = C(\xi)T(t), \quad (12)$$

where $\xi = T(t)r$, $\eta = T(t)z + c(t)$.

The forms of the perturbations $\mathbf{u} = (v_r, v_\varphi, v_z)$ and p and with the trial functions $\mathbf{f} = (f, g, h)$ and π we have:

$$\mathbf{u} = T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2 dt\right)\mathbf{f}(\xi),$$
$$p = \rho T(t)^2 \exp\left(a\eta + m\varphi + s\int T(t)^2 dt\right)\pi(\xi). \quad (13)$$

The restrictions on the forms of the basic flows following from the requirement that they satisfy Navier-Stokes equations lead to the two cases similar to those obtained for equations in Cartesian coordinates.

3 Results of the stability analysis

In this section, we discuss the fluid dynamics interpretation and stability properties of the exact solutions of the Navier-Stokes equations defined above as basic flows possessing separable stability problems.

3.1 Cartesian coordinates

Considering the class of solutions in Cartesian coordinates identified in Section 2 as Case I, we will specify the solutions by setting $c_1(t) = c_2(t) = c_3(t) = 0$ but will use a possibility to enrich the solutions by a shift of the time variable. Making change of variables $t = t' - 1/b$, where b is a constant, and omitting primes in what follows, we will have the solution of the Navier-Stokes equations in Cartesian coordinates in the form

$$V_x = \frac{1}{\sqrt{1-bt}}\left(-\frac{b\xi}{2} + \nu A(\eta)\right), V_y = \frac{b\eta}{\sqrt{1-bt}},$$
$$V_z = \frac{1}{\sqrt{1-bt}}\left(-\frac{b\zeta}{2} + \nu C(\eta)\right),$$
$$\frac{P}{\rho} = \frac{1}{1-bt}\times$$
$$\times\left(\frac{b^2}{8}\left(\xi^2 + \zeta^2 - 8\eta^2\right) - 2\nu^2\left(A_3\xi + C_3\zeta\right)\right),$$

where

$$\xi = \frac{x}{\sqrt{1-bt}},\quad \eta = \frac{y}{\sqrt{1-bt}},\quad \zeta = \frac{z}{\sqrt{1-bt}} \quad (14)$$

and b can be both positive and negative. The functions $A(\eta)$ and $C(\eta)$ are given in terms of the error $\mathrm{erf}(z)$ and generalized hypergeometric ${}_2F_2(z)$ functions. The perturbations (10) must be correspondingly specified.

The above formulas remain valid if we introduce the nondimensional variables, with the time scale $1/|b|$ and the correspondingly defined velocity scale. In the dimensionless equations (we will retain the same notation for the nondimensional variables), the parameter b takes one of the two values: $b = 1$ or $b = -1$, and ν is replaced by $1/\mathrm{Re}$ where Re is the Reynolds number. (if we mark the dimensional variables with stars, the Reynolds number will be $\mathrm{Re} = L^{*2}|b^*|/\nu^*$ where L^* is the length scale.)

We will consider the solution for the case of $b = -1$ which allows interpretations corresponding to unsteady flows near stretching (impermeable or permeable) surfaces or the flows that develop within a channel possessing permeable, moving walls. It is worth remarking that the considered flows are essentially nonparallel – the flow fields include all three velocity components dependent on all coordinates.

We choose as a criterion for stability that the ratio of the magnitude of a perturbation to that of a basic flow decreases with time, which for the solutions leads to

$$\Re\left(s+\frac{1}{2}\right)<0 \quad \text{or} \quad \Re(s)<-\frac{1}{2} \tag{15}$$

where $\Re(s)$ denotes a real part of the eigenvalue s (the imaginary part $\Im(s)$, if nonzero, determines the oscillation frequency). In particular, for the decelerating flow ($b=-1$) the meaning of instability implies that even any disturbance is damped ($\Re(s)<0$ for the velocity perturbations and $\Re(s)<1/2$ for the pressure perturbations) yet it may dominate the decelerating flow after sufficient time if $\Re(s)>-1/2$. It is also seen that the condition (15) unifies the stability criterion for the velocity and pressure perturbations.

The eigenvalue problems were solved numerically with the help of the spectral collocation method based on Chebyshev polynomials [5, 6]. For some classes of perturbations, the eigenvalue problems can be solved analytically (see below) which provides an additional, probably the most important, testing the numerical results.

It can be shown that there exists a transformation (similar in a sense to Squire's transformation [1]) such that the three-dimensional problem defined by equations (6) can be reduced to an equivalent two-dimensional problem. Then equations for the perturbation amplitudes can be reduced to a system of two equations for two functions $g(\eta)$ and $h(\eta)$ of the form

$$\begin{aligned}&\alpha\left(\alpha b-\alpha b s+\alpha^3\nu+i\nu\left(\alpha^2A(\eta)+A''(\eta)\right)\right)g(\eta)+\\&\frac{3}{2}b\alpha^2\eta g'(\eta)-\left(b-bs+2\alpha^2\nu+i\alpha\nu A(\eta)\right)g''(\eta)-\\&\frac{3}{2}b\eta g'''(\eta)+\nu g^{(\mathrm{IV})}(\eta)=0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\nu C'(\eta)g(\eta)+\left(-\frac{1}{2}b-bs+\alpha^2\nu+i\alpha\nu A(\eta)\right)h(\eta)+\\&\frac{3}{2}b\eta h'(\eta)-\nu h''(\eta)=0\end{aligned} \tag{17}$$

It is seen that for $C(\eta)=0$ the system of equations (16) and (17) decouples into two separate equations for $g(\eta)$ and $h(\eta)$. Thus, in this case two separate branches exist, first of which corresponds to the disturbances with one z-component of the velocity vector changing with x and y, while the second branch corresponds to the two-dimensional disturbances with velocity vector lying in the (x,y) plane and not dependent on z.

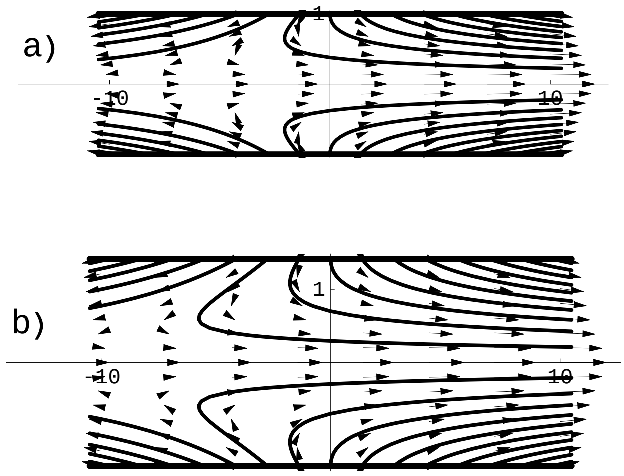

Figure 1: Flow inside an expanding porous cylinder for Re $=100$ and $U_0=5$ at different time moments: a) $t=0$; b) $t=1$.

In the case where both $A(\eta)=0$ and $C(\eta)=0$ equations (16) and (17) can be reduced to Kummer's equation [7] and can be solved in quadratures in terms of confluent hypergeometric functions.

3.2 Cylindrical coordinates

Here we will consider the class of solutions of the Navier-Stokes equations in cylindrical coordinates, which is similar in many features to the class of solutions in Cartesian coordinates considered in the previous section. The basic flow solution in cylindrical coordinates permits interpretations similar to those considered above for the solution in Cartesian coordinates. However, the cylindrical geometry and presence of the additional free parameters allow one to find more problem formulations and enrich the problem definitions. The basic flow might be again an unsteady axially symmetrical stagnation-point type flow, with the flow velocity decreasing with time as $(1+t)^{-1}$, but, as distinct from the flows considered in the previous section, here fluid flows radially from infinity approaching the axis and spreading along it. The basic flow might also be an unsteady flow inside an expanding stretching cylinder, which may also rotate, and there is an injection of fluid through the porous pipe surface. Such a flow without rotation is shown in Fig. 1.

There is an important point in which the stability problems in cylindrical coordinates differ from those in Cartesian coordinates: a transformation, similar to Squire's transformation, which reduces the three-dimensional perturbation problem to an equivalent two-dimensional problem, does not exist. Therefore, in general, one has to consider the three-dimensional perturbations

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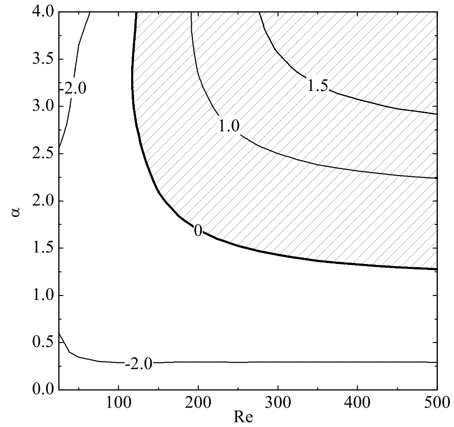

Figure 2: Neutral curve and contours of constant growth rate S for $U_0 = 30$ and $n = 2$. The shaded area represents the region in parameter space where unstable solutions exist.

to assess the flow stability. Below we present the results of numerical solution of the eigenvalue problems for the most general three-dimensional perturbations of the unsteady nonparallel flows developing within expanding pipe.

First, the analysis shows that the flow within not rotating cylinder and in the absence of the axial pressure gradient is stable ($S < 0$) in all the parameter space. All the eigenvalues are real so that the disturbances decay monotonically.

If the basic flow includes the part due to the axial pressure gradient ($U_0 \neq 0$), positive values of S appear (see Fig. 2). The neutral curve $S = 0$ in Fig. 2 separates the regions of stability and instability. It is seen that for any Reynolds number larger than some critical value Re_* (for $U_0 = 30$, $\mathrm{Re}_* \approx 120$) there exists a range of wave numbers α corresponding to unstable solutions. Thus, the flow including the part due to the axial pressure gradient is unstable for $\mathrm{Re} > \mathrm{Re}_*$. The critical Reynolds number Re_* decreases while U_0 increases.

4. Conclusions

In the present paper, we have applied the so-called direct approach to separate variables in the linear stability equations. As the result, we have defined several classes of the *exact solutions of the Navier-Stokes equations*, for which the linear stability problems are *exactly solvable*. We also determined the corresponding forms of perturbations and equations with separated variables and extended the analysis to solve numerically the eigenvalue problems for some flows. The results should help in furthering current understanding of the nonparallel flow instability physics and can provide a necessary foundation for many approximate approaches used so far. The results from exactly separable stability problems can be used for testing various assumptions and simplifications on which those theories are based.

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