Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes

Separation of variables in time-dependent Schrödinger equations

Alexander Zhalij and Renat Zhdanov

Abstract. We classify (1+3)-dimensional Schrödinger equations for a particle interacting with an electromagnetic field that are solvable by the method of separation of variables into second-order ordinary differential equations. It is established, in particular, that the necessary condition for the Schrödinger equation to be separable is that the magnetic field must be independent of the spatial variables. We describe vector-potentials that (a) provide the separability of the Schrödinger equation, (b) satisfy vacuum Maxwell equations without currents, and (c) describe a non-zero magnetic field. Furthermore, we apply the results obtained for separating variables to the Hamilton–Jacobi equation.

1. Introduction

The principal object of study in the present paper is a problem of the separation of variables in the Schrödinger equation (SE) for a particle interacting with an electromagnetic field,

$$\left(i\frac{\partial}{\partial t} - eA_0(t,\vec{x}) - \left(i\vec{\nabla} - e\vec{A}(t,\vec{x})\right)^2\right)\psi(t,\vec{x}) = 0. \tag{1.1}$$

where $A = (A_0, A_1, A_2, A_3)$ is the vector potential of the electromagnetic field, $e = \text{const}$.

As this equation has variable coefficients, a natural question arises: which equations of the form (1.1) are separable, namely, which potentials $A_0, \vec{A}$ allow for separability of the SE in some curvilinear coordinate system?

Winternitz et al. [**1, 2**] started a systematic study of potentials for which the stationary SE in two and three dimensions admits the separation of variables. This approach is based on the fact that a solution with separated variables is a common eigenfunction of first- or second-order differential operators, which commute with each other and with the operator of the equation under consideration.

This approach to separation of variables was further developed by Kalnins and Miller [**3, 4**]. See [**4**] and references contained therein for applications of this approach to equations of form (1.1).

1991 *Mathematics Subject Classification.* 35Q40, 81Q05.
Key words and phrases. separation of variables; exactly solvable and integrable systems.
The first author was supported in part by NSERC-NATO Fellowship.

©2004 American Mathematical Society

Independently, the symmetry approach to the separation of variables in the equations of quantum mechanics and quantum field theory was developed by Shapovalov [**5**] (who was the first to give a systematic treatment of the problem of variable separation in the Dirac equation using its non-Lie symmetry) and by Bagrov with collaborators [**6**]. Shapovalov and Sukhomlin [**7**] have obtained some separable SEs of the form (1.1), however, their results are not complete.

In this paper we propose an alternative approach to the problem of separation of variables in the equation (1.1). The results announced here were partially published in the papers [**8**] and [**9**].

With all the variety of approaches to the separation of variables in PDEs one can notice three general principles, namely,

a) Representation of a solution to be found in a separated (factorized) form via several functions of one variable.
b) Requirement that the above mentioned functions of one variable should satisfy some ordinary differential equations.
c) Dependence of the solution on several arbitrary (continuous or discrete) parameters, called spectral parameters, or separation constants.

By a proper formalization of the above features we will formulate an algorithm for variable separation in the SE with a vector-potential.

To have right to talk about the description of *all* potentials and *all* coordinate systems enabling us to separate the SE, one needs to provide a rigorous definition of the separation of variables. The definition we intend to use is based on ideas contained in a paper by Koornwinder [**10**].

Let us introduce a new coordinate system $\{t,\, \omega_a = \omega_a(t,\vec{x}),\, a = 1,2,3\}$, where ω_a are real-valued functions, functionally independent with respect to the spatial variables x_1, x_2, x_3:

$$\det \left\| \frac{\partial(\omega_1,\omega_2,\omega_3)}{\partial(x_1,x_2,x_3)} \right\| \neq 0. \tag{1.2}$$

For the solution to be found we adopt the following separation Ansatz:

$$\psi(t,\vec{x}) = Q(t,\vec{x})\varphi_0(t,\vec{\lambda})\varphi_1\left(\omega_1(t,\vec{x}),\vec{\lambda}\right)\varphi_2\left(\omega_2(t,\vec{x}),\vec{\lambda}\right)\varphi_3\left(\omega_3(t,\vec{x}),\vec{\lambda}\right), \tag{1.3}$$

where Q, φ_μ $(\mu = 0,1,2,3)$ are smooth functions of the indicated variables.

Definition 1.1. We say that the SE (1.1) admits separation of variables in a coordinate system $\{t,\, \omega_a = \omega_a(t,\vec{x}),\, a = 1,2,3\}$, if there exists a function $Q(t,\vec{x})$ and four ordinary differential equations

$$\varphi_0' = U_0(t,\varphi_0;\,\vec{\lambda}), \quad \varphi_a'' = U_a(\omega_a,\varphi_a,\varphi_a';\,\vec{\lambda}), \quad a = 1,2,3. \tag{1.4}$$

jointly depending on three independent parameters $\lambda_1, \lambda_2, \lambda_3$ (separation constants), such that, for each triplet $(\lambda_1,\lambda_2,\lambda_3)$ and for each set of solutions $\varphi_0(t)$, $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$, $\varphi_3(\omega_3)$ of (1.4), the function (1.3) is a solution of (1.1).

In the above formulas $U_0,\dots,U_3$ are some smooth functions of the indicated variables.

Definition 1.2. Three parameters $\lambda_1,\lambda_2,\lambda_3$ in (1.4) are called independent, if the equality

$$\operatorname{rank} \left\| \frac{\partial(U_0,U_1,U_2,U_3)}{\partial(\lambda_1,\lambda_2,\lambda_3)} \right\| = 3. \tag{1.5}$$

holds, whenever $\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)\varphi_3(\omega_3) \neq 0$.

Condition (1.5) secures the essential dependence of a solution with separated variables on the separation constants $\vec{\lambda}$.

Definition 1.1 is quite algorithmic in the sense that it contains a regular algorithm of variable separation in the SE (1.1). Formulas (1.3)–(1.5) form the input data of the method. The principal steps of the procedure of variable separation in the SE (1.1) are as follows.

(1) We insert the Ansatz (1.3) into the SE and express the derivatives φ_0', φ_1'', φ_2'', φ_3'' in terms of functions φ_0, φ_1, φ_2, φ_3, φ_1', φ_2', φ_3', using equations (1.4).
(2) We split the equality obtained with respect to the variables φ_0, φ_1, φ_2, φ_3, φ_1', φ_2', φ_3', λ_1, λ_2, λ_3, i.e. we demand that the obtained equality is transformed into identity with respect to these variables.
(3) After splitting we get an overdetermined system of nonlinear partial differential equations for the unknown functions Q, ω_1, ω_2, ω_3, A_0, A_1, A_2, A_3. Solving the system yields an exhaustive description of vector–potentials $A(t, \vec{x})$ providing separability of the SE and also the corresponding coordinate systems.

Note, that putting $Q = 1$, $\omega_a = x_a$, $a = 1, 2, 3$ in (1.3) yields the standard separation of variables in the Cartesian coordinate system. Next, choosing the spherical coordinates as $\omega_1, \omega_2, \omega_3$ we arrive at the separation of variables in the spherical coordinate system and so on. The principal task is describing all possible forms of the functions Q, ω_a, $a = 1, 2, 3$, that provide separability of the SE in the sense of the definition given above. The solution of this problem, in turn, requires describing the functions $A_0, \ldots, A_3$ that enable the separation of variables in the SE in the corresponding coordinate system. More precisely, we will need to describe *all* cases of coefficients $A_0, \ldots, A_3$, for which the corresponding SE (1.1) is separable (in the sense of Definition 1.1) in at least one coordinate system.

Note, that formulas (1.3)–(1.5) form the input data of the method. We can change these conditions and thereby modify the definition of separation of variables. For instance, we can change the order of the reduced equations (1.4), or the number of essential parameters. So, our claim of obtaining the *complete description* of vector-potentials and coordinate systems providing separation of variables in (1.1) makes sense only within the framework of Definition 1.1. If one uses a more general definition, it might be possible to construct new coordinate systems and vector–potentials providing separability of equation (1.1). However, all solutions of the SE with separated variables known to us fit into the above suggested scheme.

Next, we introduce an equivalence relation on the set of all vector–potentials $A_0(t, \vec{x})$, $\vec{A}(t, \vec{x})$ providing separability of equation (1.1), on the sets of solutions with separated variables and the corresponding coordinate systems.

Definition 1.3. We say that two vector–potentials $A(t, \vec{x})$ and $A'(t, \vec{x})$ are equivalent if they are transformed one into another by the gauge transformation

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} f, \quad A_0 \to A_0' = A_0 - \frac{\partial f}{\partial t}, \tag{1.6}$$

where $f = f(t, \vec{x})$ is an arbitrary smooth function.

For the SEs to be invariant with respect to the above transformation, the wave function $\psi(t,\vec{x})$ is to be transformed according to the rule

$$\psi \to \psi' = \psi \exp(ief) \tag{1.7}$$

Indeed, if the transformations (1.6)–(1.7) in the SE (1.1) are performed, we obtain the initial equation, provided we replace the functions $\vec{A}, A_0, \psi$ by $\vec{A}', A_0', \psi'$.

Note that the system of PDEs (1.1) admits a wider equivalence group from the point of view of the standard theory of partial differential equations [**7**]. However, this group cannot be regarded as an equivalence group within the context of quantum mechanics, since allowed transformations of the wave function must preserve the probability density $\psi^*\psi$. It is straightforward to check that the wider Shapovalov and Sukhomlin equivalence group breaks this rule, because it, generally speaking, does not preserve $\psi^*\psi$. For this reason, we restrict our considerations to gauge transformations only.

Definition 1.4. Two coordinate systems t, ω_1, ω_2, ω_3 and t', ω_1', ω_2', ω_3' are called equivalent if they give equivalent solutions with separated variables. In particular, two coordinate systems are equivalent if the corresponding Ansätze (1.3) are transformed one into another by reversible transformations of the form

$$t \to t' = f_0(t), \quad \omega_a \to \omega_a' = f_a(\omega_a), \quad a = 1,2,3, \tag{1.8}$$

$$Q \to Q' = Q l_0(t) l_1(\omega_1) l_2(\omega_2) l_3(\omega_3), \tag{1.9}$$

where $f_0, \ldots, f_3$ and $l_0, \ldots, l_3$ are some smooth functions of the indicated variables.

Indeed, transformations (1.9) preserve the form of Ansätze (1.3). Hence, after completing the procedure of separation of variables in these coordinate systems, we obtain the same solutions with separated variables.

Within these equivalence relations we can always choose the reduced equations (1.4) to be

$$i\varphi_0' = (T_0(t) - T_i(t)\lambda_i)\,\varphi_0, \quad \varphi_a'' = (F_{a0}(\omega_a) + F_{ai}(\omega_a)\lambda_i)\,\varphi_a, \tag{1.10}$$

where T_0, T_i, F_{a0}, F_{ai} are some smooth functions of the indicated variables, $a = 1,2,3$.

Having performed the first two steps of the above algorithm we obtain a system of nonlinear PDEs for the unknown functions Q, ω_1, ω_2, ω_3 in the form

$$\frac{\partial\omega_i}{\partial x_a}\frac{\partial\omega_j}{\partial x_a} = 0, \quad i \neq j, \quad i,j = 1,2,3; \tag{1.11}$$

$$\sum_{i=1}^{3} F_{ia}(\omega_i)\frac{\partial\omega_i}{\partial x_j}\frac{\partial\omega_i}{\partial x_j} = T_a(t), \quad a = 1,2,3; \tag{1.12}$$

$$2\left(\frac{\partial Q}{\partial x_j} + ieQA_j\right)\frac{\partial\omega_a}{\partial x_j} + Q\left(i\frac{\partial\omega_a}{\partial t} + \Delta\omega_a\right) = 0, \quad a = 1,2,3; \tag{1.13}$$

$$\begin{aligned} &Q\sum_{i=1}^{3} F_{i0}(\omega_i)\frac{\partial\omega_i}{\partial x_j}\frac{\partial\omega_i}{\partial x_j} + i\frac{\partial Q}{\partial t} + \Delta Q + 2ieA_a\frac{\partial Q}{\partial x_a} \\ &+Q\left(T_0(t) + ie\frac{\partial A_a}{\partial x_a} - eA_0 - e^2A_aA_a\right) = 0. \end{aligned} \tag{1.14}$$

Hereafter the summation over the repeated Latin indices from 1 to 3 is implied.

Thus the problem of variable separation in the SE reduces to integrating a system of nonlinear PDEs for eight unknown functions $A_0, A_1, A_2, A_3, Q, \omega_1, \omega_2, \omega_3$ of four variables $t, \vec{x}$. What is more, some coefficients are arbitrary functions which should be determined in the process of integrating of the system of PDEs (1.11)–(1.14). Roughly speaking, to solve a linear equation we have to solve a system of nonlinear equations.

2. Connection with the symmetry approach

Let us briefly analyze the connection between separability the SE (1.1) and its symmetry properties.

DEFINITION 2.1. The linear second order differential operator

$$L = k_{ab}(\vec{x}, t)\frac{\partial^2}{\partial x_a \partial x_b} + m_a(\vec{x}, t)\frac{\partial}{\partial x_a} + n(\vec{x}, t),$$

where k_{ab}, m_a, n, $(a, b = 1, 2, 3)$ are smooth functions of $\vec{x}, t$, is a *symmetry operator* for the SE (1.1), if

$$[L, S] = R(\vec{x}, t)S, \tag{2.1}$$

where $S = p_0 - p_a p_a$ is the operator of equation (1.1), $[L, S] = LS - SL$ is the commutator of operators L and S, and $R(\vec{x}, t)$ is some smooth function, which depends on the coefficients of the operator L.

THEOREM 2.2. *Let the SE (1.1) admit separation of variables in the sense of Definition 1.1. Then each solution with separated variables is a common eigenfunction of three mutually commuting linear second order symmetry operators of equation (1.1), the separation constants $\lambda_1, \lambda_2, \lambda_3$ being their eigenvalues.*

PROOF. Let us make the following change of variables in equation (1.1)

$$\psi(t, \vec{x}) = Q(t, \vec{x})\Psi\left(t, \omega_1(t, \vec{x}), \omega_2(t, \vec{x}), \omega_3(t, \vec{x})\right), \tag{2.2}$$

where $Q, \omega_1, \omega_2, \omega_3$ is an arbitrary solution of the system of PDE (1.11)–(1.14). Substituting the expression (2.2) into (1.1) and taking into account equations (1.11)–(1.14), we get

$$i\frac{\partial \Psi}{\partial t} + \sum_{a=1}^{3}\left(\frac{\partial^2 \Psi}{\partial \omega_a^2} - F_{a0}(\omega_a)\Psi\right)\frac{\partial \omega_a}{\partial x_c}\frac{\partial \omega_a}{\partial x_c} - T_0(t)\Psi = 0.$$

Note, that condition (1.5) and equations (1.10),(1.12) give the condition

$$\det \|F_{ab}\|_{a,b=1}^{3} \neq 0. \tag{2.3}$$

Solving equation (1.12) with respect to

$$\frac{\partial \omega_1}{\partial x_c}\frac{\partial \omega_1}{\partial x_c}, \quad \frac{\partial \omega_2}{\partial x_c}\frac{\partial \omega_2}{\partial x_c}, \quad \frac{\partial \omega_3}{\partial x_c}\frac{\partial \omega_3}{\partial x_c},$$

we have

$$\frac{\partial \omega_a}{\partial x_c}\frac{\partial \omega_a}{\partial x_c} = G_{ba}T_b, \quad a = 1, 2, 3,$$

where G_{ba} are components of the matrix, inverse to matrix $\|F_{ab}\|$. Thus, in new coordinates $t, \omega_1, \omega_2, \omega_3, \Psi(t, \omega_1, \omega_2, \omega_3)$ equation (1.1) takes the form

$$i\frac{\partial \Psi}{\partial t} + \sum_{a=1}^{3}\left(\frac{\partial^2 \Psi}{\partial \omega_a^2} - F_{a0}(\omega_a)\Psi\right)G_{ba}T_b(t) - T_0(t)\Psi = 0. \tag{2.4}$$

Now we can construct the triplet of symmetry operators. Let us consider the system of equations (1.10) for the functions $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$, $\varphi_3(\omega_3)$. After multiplying the first one by $\varphi_0\varphi_2\varphi_3$, the second one by $\varphi_0\varphi_1\varphi_3$, and the third one by $\varphi_0\varphi_1\varphi_2$ we obtain

$$\frac{\partial^2\Psi}{\partial\omega_a^2} = (F_{a0}(\omega_a) + F_{ab}(\omega_a)\lambda_b)\,\Psi, \quad a = 1,2,3. \tag{2.5}$$

In virtue of condition (2.3) we can solve the system (2.5) with respect to $\lambda_b\Psi$, $b = 1,2,3$

$$\sum_{a=1}^{3} G_{ba}\left(\frac{\partial^2\Psi}{\partial\omega_a^2} - F_{a0}(\omega_a)\Psi\right) = \lambda_b\Psi, \quad b = 1,2,3,$$

The solution with separated variables

$$\Psi = \varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)\varphi_3(\omega_3)$$

is a common eigenfunction of the three operators

$$P_a = \sum_{b=1}^{3} G_{ab}\left(\frac{\partial^2}{\partial\omega_b^2} - F_{b0}(\omega_b)\right), \quad a = 1,2,3, \tag{2.6}$$

(G_{ab} is matrix, inverse to $\|F_{ab}(\omega_a)\|$) and moreover the equality

$$P_a\Psi = \lambda_a\Psi. \tag{2.7}$$

holds.

By a direct (and very cumbersome) computation one can check, that the second order differential operators P_1, P_2, P_3 commute pairwise for arbitrary functions $F_{ab}(\omega_a)$, $F_{a0}(\omega_a)$, $a, b = 1,2,3$, i.e.

$$[P_a, P_b] = P_aP_b - P_bP_a = 0, \quad a, b = 1,2,3. \tag{2.8}$$

After being rewritten in terms of the operators P_1, P_2, P_3 equation (2.4) reads

$$i\frac{\partial\Psi}{\partial t} + T_a(t)P_a\Psi - T_0(t)\Psi = 0.$$

Since the relations

$$[i\frac{\partial}{\partial t} + T_a(t)P_a - T_0(t), P_a] = 0, \quad a = 1,2,3, \tag{2.9}$$

hold, operators P_1, P_2, P_3 are mutually commuting symmetry operators for equation (2.4). If we designate by P_1', P_2', P_3' the operators P_1, P_2, P_3 written in the initial variables $t, \vec{x}, \psi$, then from (2.7)–(2.9) we get the following equalities

$$[p_0 - p_cp_c, P_a'] = 0, \quad P_a'\psi = \lambda_a\psi, \quad [P_a', P_b'] = 0, \quad a, b = 1,2,3,$$

where $\psi = Q(t,\vec{x})\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)\varphi_3(\omega_3)$, and $p_0 - p_cp_c$ is operator of equation (1.1). □

Thus there are two possible approaches to variable separation in linear PDEs which are based on their symmetry properties. The first one is to start with a set of commuting symmetry operators of the equation under study and to finish by constructing the coordinate systems (Winternitz at al [**1, 2**], Kalnins and Miller [**3, 4**], Shapovalov [**5, 7**]).

Another approach, that proposed in papers [**8, 9**], is closer to the original understanding of the separation of variables in PDE. A desired form (1.3) of a solution with separated variables is postulated and then it turns out that the solution obtained can be related to a set of mutually commuting symmetry operators of the equations under consideration.

Both approaches have their merits and drawbacks. We think that the utilization of the first approach is the only way to separate variables in multicomponent systems of PDEs. But to separate variables with one dependent variable it is preferable to apply the second approach, since a computation of symmetry operators is an extra step which is not, in fact, necessary for obtaining solution with separated variables.

3. Main results

We do not give the computations needed for integrating of the system of PDEs (1.11)–(1.14) in full detail, because they are very cumbersome. The details can be found in the paper [**8**].

Integration of the system (1.11)–(1.12) gives the most general form of separable coordinate systems. Its general form $\vec{\omega} = \vec{\omega}(t, \vec{x})$ is given implicitly within the equivalence relation (1.8) by the following formulas:

$$\vec{x} = \mathcal{O}(t)\mathcal{L}(t)\left(\vec{z}(\vec{\omega}) + \vec{v}(t)\right), \tag{3.1}$$

Here $\mathcal{O}(t)$ is a time-dependent 3×3 orthogonal matrix with Euler angles $\alpha(t), \beta(t), \gamma(t)$:

$$\mathcal{O}(t) = \begin{pmatrix} \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta\cos\gamma & -\cos\alpha\,\sin\beta - \sin\alpha\,\cos\beta\cos\gamma & \sin\alpha\,\sin\gamma \\ \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta\cos\gamma & -\sin\alpha\,\sin\beta + \cos\alpha\,\cos\beta\cos\gamma & -\cos\alpha\,\sin\gamma \\ \sin\beta\,\sin\gamma & \cos\beta\,\sin\gamma & \cos\gamma \end{pmatrix};$$

$\vec{v}(t)$ stands for the vector-column whose entries $v_1(t), v_2(t), v_3(t)$ are arbitrary smooth functions of t; $\vec{z} = \vec{z}(\vec{\omega})$ is given by one of the eleven formulas

(1) Cartesian coordinate system,

$$z_1 = \omega_1, \quad z_2 = \omega_2, \quad z_3 = \omega_3,$$
$$\omega_1, \omega_2, \omega_3 \in \mathbf{R}.$$

(2) Cylindrical coordinate system,

$$z_1 = e^{\omega_1}\cos\omega_2, \quad z_2 = e^{\omega_1}\sin\omega_2, \quad z_3 = \omega_3,$$
$$0 \le \omega_2 < 2\pi, \quad \omega_1, \omega_3 \in \mathbf{R}.$$

(3) Parabolic cylindrical coordinate system,

$$z_1 = (\omega_1^2 - \omega_2^2)/2, \quad z_2 = \omega_1\omega_2, \quad z_3 = \omega_3,$$
$$\omega_1 > 0, \quad \omega_2, \omega_3 \in \mathbf{R}.$$

(4) Elliptic cylindrical coordinate system,

$$z_1 = a\cosh\omega_1\cos\omega_2, \quad z_2 = a\sinh\omega_1\sin\omega_2, \quad z_3 = \omega_3,$$
$$\omega_1 > 0, \quad -\pi < \omega_2 \le \pi, \quad \omega_3 \in \mathbf{R}, \quad a > 0.$$

(5) Spherical coordinate system,

$$z_1 = \omega_1^{-1}\,\mathrm{sech}\,\omega_2\cos\omega_3,$$
$$z_2 = \omega_1^{-1}\,\mathrm{sech}\,\omega_2\sin\omega_3,$$
$$z_3 = \omega_1^{-1}\tanh\omega_2,$$
$$\omega_1 > 0, \quad \omega_2 \in \mathbf{R}, \quad 0 \le \omega_3 < 2\pi.$$

(6) Prolate spheroidal coordinate system,

$$\begin{aligned} z_1 &= a\,\mathrm{csch}\,\omega_1\,\mathrm{sech}\,\omega_2\cos\omega_3, \quad a>0,\\ z_2 &= a\,\mathrm{csch}\,\omega_1\,\mathrm{sech}\,\omega_2\sin\omega_3,\\ z_3 &= a\coth\omega_1\tanh\omega_2,\\ &\omega_1>0, \quad \omega_2\in\mathbf{R}, \quad 0\le\omega_3<2\pi. \end{aligned}$$

(7) Oblate spheroidal coordinate system,

$$\begin{aligned} z_1 &= a\csc\omega_1\,\mathrm{sech}\,\omega_2\cos\omega_3, \quad a>0,\\ z_2 &= a\csc\omega_1\,\mathrm{sech}\,\omega_2\sin\omega_3,\\ z_3 &= a\cot\omega_1\tanh\omega_2,\\ &0<\omega_1<\pi/2, \quad \omega_2\in\mathbf{R}, \quad 0\le\omega_3<2\pi. \end{aligned}$$

(8) Parabolic coordinate system,

$$\begin{aligned} z_1 &= e^{\omega_1+\omega_2}\cos\omega_3, \quad z_2 = e^{\omega_1+\omega_2}\sin\omega_3,\\ z_3 &= (e^{2\omega_1}-e^{2\omega_2})/2,\\ &\omega_1,\omega_2\in\mathbf{R}, \quad 0\le\omega_3\le 2\pi. \end{aligned}$$

(9) Paraboloidal coordinate system,

$$\begin{aligned} z_1 &= 2a\cosh\omega_1\cos\omega_2\sinh\omega_3, \quad a>0,\\ z_2 &= 2a\sinh\omega_1\sin\omega_2\cosh\omega_3,\\ z_3 &= a(\cosh 2\omega_1+\cos 2\omega_2-\cosh 2\omega_3)/2,\\ &\omega_1,\omega_3\in\mathbf{R}, \quad 0\le\omega_2<\pi. \end{aligned}$$

(10) Ellipsoidal coordinate system,

$$\begin{aligned} z_1 &= a\,\frac{1}{\mathrm{sn}(\omega_1,\,k)}\,\mathrm{dn}(\omega_2,\,k')\,\mathrm{sn}(\omega_3,\,k), \quad a>0,\\ z_2 &= a\,\frac{\mathrm{dn}(\omega_1,\,k)}{\mathrm{sn}(\omega_1,\,k)}\,\mathrm{cn}(\omega_2,\,k')\,\mathrm{cn}(\omega_3,\,k),\\ z_3 &= a\,\frac{\mathrm{cn}(\omega_1,\,k)}{\mathrm{sn}(\omega_1,\,k)}\,\mathrm{sn}(\omega_2,\,k')\,\mathrm{dn}(\omega_3,\,k),\\ &0<\omega_1<K, \quad -K'\le\omega_2\le K', \quad 0\le\omega_3\le 4K. \end{aligned}$$

(11) Conical coordinate system,

$$\begin{aligned} z_1 &= \omega_1^{-1}\mathrm{dn}(\omega_2,\,k')\,\mathrm{sn}(\omega_3,\,k),\\ z_2 &= \omega_1^{-1}\mathrm{cn}(\omega_2,\,k')\,\mathrm{cn}(\omega_3,\,k),\\ z_3 &= \omega_1^{-1}\mathrm{sn}(\omega_2,\,k')\,\mathrm{dn}(\omega_3,\,k),\\ &\omega_1>0, \quad -K'\le\omega_2\le K', \quad 0\le\omega_3\le 4K. \end{aligned}$$

and $\mathcal{L}(t)$ is a 3×3 diagonal matrix

$$\mathcal{L}(t) = \begin{pmatrix} l_1(t) & 0 & 0\\ 0 & l_2(t) & 0\\ 0 & 0 & l_3(t) \end{pmatrix}, \tag{3.2}$$

where $l_1(t), l_2(t), l_3(t)$ are arbitrary non-zero smooth functions that satisfy the following conditions

- $l_1(t) = l_2(t)$ for the partially split coordinate systems (cases 2–4 in the above list)),
- $l_1(t) = l_2(t) = l_3(t)$ for non-split coordinate systems (cases 5–11 in in the above list)).

Note that we have chosen the coordinate systems $\omega_1, \omega_2, \omega_3$ by means of the equivalence relation (1.8) in such a way that the relations

$$\Delta\omega_a = 0, \quad a = 1, 2, 3 \tag{3.3}$$

hold for all eleven coordinate systems.

From a geometric point of view the right-hand side of formula (3.1) is a result of the application to vector $\vec{z}(\vec{\omega})$ of the following time-dependent transformations performed one after another:

(1) translations $\vec{z} \to \vec{z}' = \vec{z} + \vec{v}(t)$,
(2) dilatations $\vec{z} \to \vec{z}' = \mathcal{L}(t)\vec{z}$,
(3) three-dimensional rotations $\vec{z} \to \vec{z}' = \mathcal{O}(t)\vec{z}$ with Euler angles $\alpha(t)$, $\beta(t)$, $\gamma(t)$.

Together with the rotations the following vector $\vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3)$ is considered

$$\begin{aligned}\Omega_1(t) &= \dot{\gamma}(t)\cos\alpha(t) + \dot{\beta}(t)\sin\alpha(t)\sin\gamma(t),\\ \Omega_2(t) &= \dot{\gamma}(t)\sin\alpha(t) - \dot{\beta}(t)\cos\alpha(t)\sin\gamma(t),\\ \Omega_3(t) &= \dot{\alpha}(t) + \dot{\beta}(t)\cos\gamma(t),\end{aligned} \tag{3.4}$$

that is directed along the instantaneous axis of rotation and is called the *angular velocity vector*.

After integrating system (1.11)–(1.12) it is not difficult to integrate with the use of (3.3) the remaining equations (1.13)–(1.14) in the system under study. They can be regarded as algebraic equations for the functions $A_a(t, \vec{x}), (a = 1, 2, 3)$ and $A_0(t, \vec{x})$, correspondingly.

THEOREM 3.1. *The SE (1.1) admits separation of variables (in the sense of definition 1.1) if and only if it is gauge equivalent to a SE where*

– *the magnetic field $\vec{H} = \mathrm{rot}\,\vec{A}$ is independent of the spatial variables,*
– *the space–like components A_1, A_2, A_3 of the vector–potential of the electromagnetic field are given by*

$$\vec{A}(t, \vec{x}) = \frac{1}{2}\begin{pmatrix} 0 & -H_3(t) & H_2(t) \\ H_3(t) & 0 & -H_1(t) \\ -H_2(t) & H_1(t) & 0 \end{pmatrix}\vec{x} = \frac{1}{2}\vec{H}(t) \times \vec{x}, \tag{3.5}$$

where the symbol $\times$ denotes the vector product,
– *the time–like component A_0 is given by formula*

$$eA_0(t, \vec{x}) = \sum_{a=1}^{3} F_{a0}(\omega_a)\frac{\partial\omega_a}{\partial x_b}\frac{\partial\omega_a}{\partial x_b} - T_0(t) - e^2 A_b A_b - \frac{1}{4}P. \tag{3.6}$$

Here T_0, F_{a0} are some smooth functions of the indicated variables, and P is some second-order polynomial in the new variables $\vec{\omega}$ with coefficients that are functions of t (for more details see [**9**]).

Furthemore, we have proved the following theorem:

THEOREM 3.2. *Let the SE (1.1) admit separation of variables in some non-stationary coordinate system* t, $\omega_a = \omega_a(t,\vec{x})$, $a = 1,2,3$, *where the functions* $\omega_1(t,\vec{x})$, $\omega_2(t,\vec{x})$, $\omega_3(t,\vec{x})$ *are given implicitly by formulas (3.1)–(3.2). Then the angular velocity vector (3.4) of the rotation of this coordinate system equals* $-e\vec{H}$, *where* $\vec{H} = \mathrm{rot}\,\vec{A}$ *is magnetic field.*

It follows from the last theorem that a necessary condition for the SE (1.1) with non-zero magnetic field $\vec{H}$ to be separable (in the sense of our Definition 1.1) is that the angular velocity vector (3.4) of rotation of the separation coordinate system (3.1)–(3.2) has to be non-zero.

We have proven, that the magnetic field $\vec{H} = \mathrm{rot}\,\vec{A}$ has to be independent of the spatial variables. Next, we have eleven classes of potentials $A_0(t,\vec{x})$, corresponding to eleven classes of coordinate systems $t, \omega_a = \omega_a(t,\vec{x})$, $a = 1,2,3$, where the functions $\omega_1(t,\vec{x}), \omega_2(t,\vec{x}), \omega_3(t,\vec{x})$ are given implicitly by formulas (3.1)–(3.2).

We will finish this section with the following remark. It follows from Theorem 3.1 that the choice of magnetic fields $\vec{H}$ allowing for variable separation in the corresponding SE is very restricted. Namely, the magnetic field should be independent of the spatial variables x_1, x_2, x_3 in order to provide the separability of the SE (1.1) into three second-order ordinary differential equations of the form (1.4). However, if we allow for the separation equations to be of a lower order, then additional possibilities for variable separation in the SE arise. As an example, we give the vector potential

$$A(t,\vec{x}) = \left(A_0\left(\sqrt{x_1^2+x_2^2}\right),\ 0,\ 0,\ A_3\left(\sqrt{x_1^2+x_2^2}\right)\right),$$

where A_0, A_3 are arbitrary smooth functions. The SE (1.1) with this vector-potential separates in the cylindrical coordinate system

$$t, \quad \omega_1 = \ln\left(\sqrt{x_1^2+x_2^2}\right), \quad \omega_2 = \arctan(x_1/x_2), \quad \omega_3 = x_3$$

into two first-order and one second-order ordinary differential equations. The corresponding magnetic field $\vec{H} = \mathrm{rot}\,\vec{A}$ is evidently x-dependent. In this respect, let us also mention the recent paper by Benenti at al. [**11**], where the problem of separation of variables in the stationary Hamilton-Jacobi equation with vector-potential has been studied. They have presented a number of vector-potentials, for which the Hamilton-Jacobi equation is separable, and the corresponding magnetic fields are inhomogeneous ones. These potentials allow for separation of variables in the stationary SEs with vector-potentials as well (in the next section we will point out the relationship between the separation of variables in the Schrödinger and Hamilton-Jacobi equations). These facts, as well as paper [**7**], imply that it is important to apply our approach to classify the non-stationary SEs of the form (1.1), which admit separation of variables into first- and second-order ordinary differential equations. Here we give the classification results for the case, when all the reduced equations are second-order ones. We intend to address this problem in a future publication.

4. Separation of variables in the Hamilton–Jacobi equation

It is well known [**12**] that there exists a deep connection between the separation of variables in the Schrödinger and Hamilton–Jacobi equations. The Hamilton–Jacobi equation,

$$u_t + eA_0 + (u_{x_a} + eA_a)(u_{x_a} + eA_a) = 0, \tag{4.1}$$

separates in any coordinate system providing separability of the Schrödinger equations (1.1) and, what is more, the inverse assertion is not true. We will make use of this connection in order to classify separable Hamilton–Jacobi equations.

First we fix the usual form of the separation Ansatz for the Hamilton–Jacobi equation,

$$u(t,\vec{x}) = S(t,\vec{x}) + \varphi_0(t) + \sum_{i=1}^{3} \varphi_i(\omega_i(t,\vec{x})), \tag{4.2}$$

and, furthermore, fix the form of the ordinary differential equations for $\varphi_0, \varphi_1, \varphi_2, \varphi_3$,

$$\varphi_0' = -T_0(t) - T_i(t)\lambda_i, \quad \varphi_a' = \left(-F_{a0}(\omega_a) + F_{ai}(\omega_a)\lambda_i\right)^{1/2}. \tag{4.3}$$

Inserting the Ansatz (4.2) into equation (4.1), eliminating the first derivatives of the functions $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ with the use of the above equations and splitting with respect to the variables $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \lambda_1, \lambda_2, \lambda_3$, we arrive at the following system of nonlinear partial differential equations for the functions $S, \omega_1, \omega_2, \omega_3$:

$$\begin{aligned}
&\frac{\partial \omega_i}{\partial x_a}\frac{\partial \omega_j}{\partial x_a} = 0, \quad i \neq j, \quad i, j = 1, 2, 3;\\
&\sum_{i=1}^{3} F_{ia}(\omega_i)\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} = T_a(t), \quad a = 1, 2, 3;\\
&2\left(\frac{\partial S}{\partial x_j} + eA_j\right)\frac{\partial \omega_a}{\partial x_j} + \frac{\partial \omega_a}{\partial t} = 0, \quad a = 1, 2, 3;\\
&-\sum_{i=1}^{3} F_{i0}(\omega_i)\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} + \frac{\partial S}{\partial t} + 2eA_a\frac{\partial S}{\partial x_a} + \frac{\partial S}{\partial x_a}\frac{\partial S}{\partial x_a}\\
&\quad -T_0(t) + eA_0 + e^2 A_a A_a = 0.
\end{aligned} \tag{4.4}$$

It is not difficult to check that the change of variables,

$$Q(t,\vec{x}) = \exp(iS(t,\vec{x})), \tag{4.5}$$

in (1.11)–(1.14) yields a system that coincides with (4.4) with the exception of the last equation, where an additional term $-i(\Delta S + eA_{ax_a})$ appears. But this term is a function of t only and is absorbed by T_0. Consequently, all the results on variable separation for the SE apply to the case of the Hamilton–Jacobi equation (4.1) as well.

5. Separation of variables in the Schrödinger-Maxwell system

The expressions (3.5), (3.6) give the most general form of the vector-potential of the electromagnetic field, providing separability of the corresponding SEs. In view of the generality of the results, these expressions are cumbersome, and their physical interpretation is somewhat difficult. Therefore, it would be interesting to know the form of these potentials under certain physical restrictions. The most natural restriction is that the vector-potential satisfies the vacuum Maxwell equations without currents.

In this section we describe all explicit forms of the vector-potentials $A(t,\vec{x})$ that

a) Allow separability of the SE,
b) Satisfy vacuum Maxwell equations without currents,
c) Describe the non-zero magnetic field.

Omitting the details of the calculations we present below the results. Note, when presenting lists of the vector-potentials $A(t,\vec{x})$ we use invariance of the system of the Schrödinger and Maxwell equations with respect to the groups of rotations by spatial variables x_1, x_2, x_3 and translations of the all variables t, x_1, x_2, x_3.

Case of non-stationary magnetic field.

$$\begin{aligned} e\vec{H} &= (0, 0, At + B)\,, \\ eA_0 &= -\frac{k}{2}(x_1^2 + x_2^2 - 2x_3^2) + a_1x_1 + a_2x_2 + a_3x_3, \end{aligned}$$

where A, B, k, a_1, a_2, a_3 are arbitrary real constants.

The coordinate system is

$$\vec{x} = \mathcal{L}\mathcal{O}(\vec{z} + \vec{v}).$$

Here $\mathcal{O}$ is a time-dependent 3×3 orthogonal matrix $\mathcal{O}(\alpha, \beta, \gamma)$, where

$$\alpha = -\frac{1}{2}At^2 - Bt, \quad \beta = 0, \quad \gamma = 0;$$

$\vec{z}$ is the cartesian, cylindrical or elliptic cylindrical coordinates; $\mathcal{L}$ is the 3×3 diagonal matrix

$$\mathcal{L} = \begin{pmatrix} l(t) & 0 & 0 \\ 0 & l(t) & 0 \\ 0 & 0 & l_3(t) \end{pmatrix},$$

and $\vec{v}(t)$ is vector-column $\vec{v}(t) = (v_1, v_2, v_3)^T$ where functions $l(t), l_3(t), v_1(t), v_2(t), v_3(t)$ are solutions of the following system of ordinary differential equations:

$$\begin{aligned} &2\frac{c}{l^4} - \frac{1}{2}\frac{\ddot{l}}{l} + k = \frac{1}{2}(At + B)^2, \quad \frac{c_3}{l_3^4} - \frac{1}{4}\frac{\ddot{l}_3}{l_3} = k, \\ &l\ddot{v}_1 + 2\dot{l}\dot{v}_1 + 4c\frac{v_1}{l^3} - 2c_{11}\frac{1}{l} = -2(a_1\cos\alpha + a_2\sin\alpha), \\ &l\ddot{v}_2 + 2\dot{l}\dot{v}_2 + 4c\frac{v_2}{l^3} - 2c_{12}\frac{1}{l} = -2(-a_1\sin\alpha + a_2\cos\alpha), \\ &l_3\ddot{v}_3 + 2\dot{l}_3\dot{v}_3 + 4c_3\frac{v_3}{l_3^3} - 2c_{13}\frac{1}{l_3} = -2a_3. \end{aligned}$$

Here $c, c_3, c_{11}, c_{12}, c_{13}$ are arbitrary real constants.

Cases of stationary magnetic field.

Case 1:

$$e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0;$$
$$eA_0 = -\frac{k^2}{12}(x_1^2 + x_2^2 - 2x_3^2) + a_1x_1 + a_2x_2 + a_3x_3,$$

where $\vec{a} = (a_1, a_2, a_3)$ is constant vector.

The coordinate system is

$$\vec{x} = l\mathcal{O}(\vec{z} + \vec{v}).$$

Here $\mathcal{O}$ is a time-dependent 3×3 orthogonal matrix $\mathcal{O}(\alpha, \beta, \gamma)$, where $\alpha = -kt, \beta = \text{const}, \gamma = \text{const}$; $\vec{z}$ is one of eleven coordinate systems, listed above; function $l(t)$ is solution of the equation

$$k^2 + \frac{3}{2}\frac{\ddot{l}}{l} = \frac{c}{l^4}$$

given by one of the formulas:

$$c = \mp 1, \quad l^2 = \sqrt{C_1^2 \pm \frac{1}{k^2}} \sin\left(2\sqrt{\frac{2}{3}}kt\right) + C_1,$$

for all coordinate systems with the exception of parabolic cylindrical and parabolic ones and

$$c = 0, \quad l = C_1 \sin\left(\sqrt{\frac{2}{3}}kt\right)$$

for all eleven coordinate systems. Here C_1 is an arbitrary real constant. Vector $\vec{v}$ is a solution of the following system of ordinary differential equations:

$$3l\ddot{\vec{v}} + 6\dot{l}\dot{\vec{v}} + \frac{2c}{l^3}\vec{v} = -6\mathcal{O}^{-1}\vec{a}.$$

Case 2:

$$e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0;$$
$$eA_0 = \frac{a}{\sqrt{x_1^2 + x_2^2 + x_3^2}} - \frac{k^2}{12}(x_1^2 + x_2^2 - 2x_3^2), \quad a = \text{const} \neq 0.$$

The coordinate system is

$$\vec{x} = \mathcal{O}\vec{z}.$$

Here $\mathcal{O}$ is a time-dependent 3×3 orthogonal matrix $\mathcal{O}(\alpha, \beta, \gamma)$, where $\alpha = -kt, \beta = \text{const}, \gamma = \text{const}$ and $\vec{z}$ is one of the following coordinate systems: spherical, prolate spheroidal II (where one should replace z_3 by $z_3 = a(\coth\omega_1 \tanh\omega_2 \pm 1)$), and conical.

Case 3:

$$e\vec{H} = (0, 0, k), \quad k = \text{const} \neq 0;$$
$$eA_0 = -\frac{k^2}{12}(x_1^2 + x_2^2 - 2x_3^2) + \frac{a_1}{r} + a_2\frac{x_3}{r^3} + \frac{a_3}{r^2}\left(\frac{x_3}{2r}\ln\frac{r + x_3}{r - x_3} - 1\right),$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and a_1, a_2, a_3 are real constant numbers.

The coordinate system is

$$\vec{x} = l\mathcal{O}\vec{z}.$$

Here $\mathcal{O}$ is a time-dependent 3×3 orthogonal matrix $\mathcal{O}(\alpha,\beta,\gamma)$, where $\alpha=-kt$, $\beta=\gamma=0$; $\vec{z}$ is the spherical coordinates and function $l(t)$ is given by

$$l^2=\sqrt{C_1^2\pm\frac{1}{k^2}}\sin\left(2\sqrt{\frac{2}{3}}kt\right)+C_1,\quad \text{or}\quad l=C_1\sin\left(\sqrt{\frac{2}{3}}kt\right)$$

under condition $a_1=0$ and $l=1$ under condition $a_1\neq 0$. Here C_1 is an arbitrary real constant.

Case 4:

$$e\vec{H}=(0,0,k),\quad k=\text{const}\neq 0;$$
$$eA_0=-\frac{k^2}{12}(x_1^2+x_2^2-2x_3^2)+\frac{a_1}{r^+}+\frac{a_2}{r^-}+a_3\left(\frac{1}{r^+}\,\text{arctanh}\frac{x_3^+}{r^+}-\frac{1}{r^-}\,\text{arctanh}\frac{x_3^-}{r^-}\right),$$

where $x_3^\pm=x_3\pm a$ and $r^\pm=\sqrt{x_1^2+x_2^2+(x_3\pm a)^2}$, and a,a_1,a_2,a_3 are arbitrary real constants. The coordinate system is

$$\vec{x}=\mathcal{O}\vec{z}.$$

Here $\mathcal{O}$ is a time-dependent 3×3 orthogonal matrix $\mathcal{O}(\alpha,\beta,\gamma)$, where $\alpha=-kt$, $\beta=\gamma=0$ and $\vec{z}$ is the prolate spheroidal coordinates.

Case 5:

$$e\vec{H}=(0,0,k),\quad k=\text{const}\neq 0;$$
$$eA_0=-\frac{k^2}{12}(x_1^2+x_2^2-2x_3^2)+2a_1a\frac{f_1}{f}+2a_2\frac{x_3}{ff_1}-2a_3\left(a\frac{f_1}{f}\,\text{arccot}f_1-\frac{x_3}{ff_1}\,\text{arctanh}\frac{x_3}{af_1}\right),$$

where

$$f=\sqrt{(a^2-r^2)^2+4a^2x_3^2},\quad f_1=\sqrt{\frac{-a^2+r^2+f}{2a^2}},\quad r=\sqrt{x_1^2+x_2^2+x_3^2},$$

and a,a_1,a_2,a_3 are arbitrary real constants. The coordinate system is

$$\vec{x}=\mathcal{O}\vec{z}.$$

Here $\mathcal{O}$ is a time-dependent 3×3 orthogonal matrix $\mathcal{O}(\alpha,\beta,\gamma)$, where $\alpha=-kt$, $\beta=\gamma=0$ and $\vec{z}$ is the oblate spheroidal coordinates.

Note that expression for A_0 can be rewritten in the form

$$eA_0=-\frac{k^2}{12}(x_1^2+x_2^2-2x_3^2)+\frac{a_1+ia_2}{\tilde{r}^+}+\frac{a_1-ia_2}{\tilde{r}^-}+ia_3\left(\frac{1}{\tilde{r}^+}\,\text{arctanh}\frac{\tilde{x}_3^+}{\tilde{r}^+}-\frac{1}{\tilde{r}^-}\,\text{arctanh}\frac{\tilde{x}_3^-}{\tilde{r}^-}\right),$$

where $\tilde{x}_3^\pm=x_3\pm ia$ and $\tilde{r}^\pm=\sqrt{x_1^2+x_2^2+(x_3\pm ia)^2}$.

Case 6:

$$e\vec{H}=(0,0,k),\quad k=\text{const}\neq 0;$$
$$eA_0=-\frac{k^2}{6}(x_1^2+x_2^2-2x_3^2)+\frac{a_1}{r}+a_2x_3+\frac{a_3}{r}\ln\frac{r+x_3}{r-x_3},$$

where $r=\sqrt{x_1^2+x_2^2+x_3^2}$ and a_1,a_2,a_3 are arbitrary real constants.

The coordinate system is

$$\vec{x}=\mathcal{O}\vec{z}.$$

Here $\mathcal{O}$ is a time-dependent 3×3 orthogonal matrix $\mathcal{O}(\alpha,\beta,\gamma)$, where $\alpha=-kt$, $\beta=\gamma=0$ and $\vec{z}$ is the parabolic coordinates.

Case 7:

$$e\vec{H} = (0,0,k), \quad k = \text{const} \neq 0;$$
$$eA_0 = -\frac{q}{2}(x_1^2 + x_2^2 - 2x_3^2) + a\ln(x_1 + x_2) + a_3 x_3,$$

where k, a, a_3 are arbitrary real constants.

The coordinate system is

$$x_1 = e^{\omega_1}\cos(\omega_1 - kt), \quad x_2 = e^{\omega_1}\sin(\omega_1 - kt), \quad x_3 = l_3\omega_3 + v_3,$$

where l_3, v_3 are solutions of the system of ordinary differential equations

$$\frac{c_3}{l_3^4} - \frac{1}{4}\frac{\ddot{l}_3}{l_3} = q, \quad l_3\ddot{v}_3 + 2\dot{l}_3\dot{v}_3 + 4c_3\frac{v_3}{l_3^3} - 2c_{13}\frac{1}{l_3} = -2a_3,$$

where c_3 and c_{13} are arbitrary real constants.

Case of non-stationary magnetic field and cases 1-2 of stationary magnetic field allow for separability of the SE in more then one coordinate systems, and these cases are superintegrable ones. The corresponding sets of second-order symmetry operators can be constructed in the explicit form with the help of the theorem 2.2.

Note that some of the potentials obtained have a clear physical meaning. For instance, cases 2 and 3 under condition $k = a_2 = a_3 = 0$ give the standard Coulomb potential. Case 4 under condition $k = a_3 = 0$ gives the potential for a well-known two-center Kepler problem, i.e., the problem of finding wave functions of an electron moving in the field of two fixed Coulomb centres with charges a_1, a_2 and intercenter distance $2a$ (the model of a ionized hydrogen molecule). Coulson and Joseph [**13**] showed that the corresponding Schrödinger equation admits separation of variables in the prolate coordinate system only. We obtained this potential as a particular case of the more general potential.

Acknowledgments. This paper was finished while one of the authors (AZ) was staying at the Centre de recherches mathématiques (Université de Montréal) as an NSERC-NATO Fellow. It is a pleasure for him to thank prof. Pavel Winternitz for hospitality and very useful comments.

References

[1] P. Winternitz, J. A. Smorodinsky, M. Uhliř, and I. Friš, *Symmetry groups in classical and quantum mechanics*, Yad. Fiz. **4**, 625 (1966) [Sov. J. Nucl. Phys. **4**, 444 (1966)].

[2] A. A. Makarov, J. A. Smorodinsky, Kh. Valiev, and P. Winternitz, *Systematic search for nonrelativistic systems with dynamical symmetries*, Nuovo Cimento A **52**, 1061 (1967).

[3] W. Miller, Jr., *Symmetry and Separation of Variables*, Addison-Wesley, Reading, MA, 1977.

[4] E. G. Kalnins and W. Miller, Jr., *R-separation of variables for the time-dependent Hamilton-Jacobi and Schroedinger equations*, J. Math. Phys., **28** (1987), pp. 1005-1015.

[5] Shapovalov, V.N., *Separation of variables in second-order linear differential equations*, Differetsial'nye uravneniya, **16** (1980), No. 10, 1864—1874. [Differ. Equations **16** (1981), 1212-1220].

[6] V. G. Bagrov and D. M. Gitman, *Exact Solutions of Relativistic Wave Equations.* Mathematics and its Applications (Soviet Series), 39. Kluwer Academic Publishers, Dordrecht, 1990.

[7] V. N. Shapovalov and N. B. Sukhomlin, *Separation of variables in the nonstationary Schrödinger equation*, Izv. Vyssh. Uchebn. Zaved., Fiz. no. 12, 100–105 (1974) [Sov. Phys. J. **17**, 1718 (1976)].

[8] R. Zhdanov and A. Zhalij, *On separable Schrödinger equations*, J. Math. Phys. **40**, no. 12, 6319–6338 (1999).

[9] A. Zhalij, *On separable Pauli equations*, J. Math. Phys. **43**, no. 3, 1365–1389 (2002).

[10] T. H. Koornwinder, *A precise definition of separation of variables*, Lecture Notes in Math. **810**, 240–263 (1980).

[11] S. Benenti, C. Chanu, and G. Rastelli, *Variable separation for natural Hamiltonians with scalar and vector potentials on Riemannian manifolds*, J. Math. Phys. **42**, no. 5, 2065–2091 (2001)

[12] S. Benenti, C. Chanu, and G. Rastelli, *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrodinger equation*: I. The completeness and the Robertson conditions, J. Math. Phys. **43**, no. 10, 5183-5222 (2002); II. First integrals and symmetry operators, J. Math. Phys. **43**, no. 10, 5223-5253 (2002).

[13] C. A. Coulson and A. Joseph, *Constant of the motion for the two-centre Kepler problem*, Intern. J. Quant. Chem. **1**, 337–347 (1967).

Institute of Mathematics of the Academy of Sciences of Ukraine, Tereshchenkivska Street 3, 01601 Kyiv-4, Ukraine

Current address: Centre de recherches mathématiques, C.P. 6128 succursale Centre-ville, Montréal (Québec) H3C 3J7, Canada

E-mail address: zhaliy@imath.kiev.ua, zhalij@crm.umontreal.ca

Institute of Mathematics of the Academy of Sciences of Ukraine, Tereshchenkivska Street 3, 01601 Kyiv-4, Ukraine

E-mail address: renat@imath.kiev.ua