The axisymmetrical poroelasticity problem for a solid cylinder with a thin interface

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The poroelastic cylinder $(0 < r < 1, 0 < z < h, -\pi < \varphi < \pi)$ is considered in terms of Biot's model [1]. It consists of two main parts $\Omega_- = \{0 < r < 1, 0 < z < d - h_0/2\}$, $\Omega_+ = \{0 < r < 1, d + h_0/2 < z < h\}$ and a thin interlayer $\Omega_0 = \{0 < r < 1, d - h_0/2 < z < d + h_0/2\}$ between them. Here $h_0 \ll d$. Between each pair Ω_- and Ω_0, Ω_0 and Ω_+ the ideal contact conditions are fulfilled [2].

The cylindrical surface $r = 1$ is impermeable under ideal contact conditions [2] $u(1, z) = 0, \tau_{rz}(1, z) = 0, \frac{\partial p}{\partial r}(1, z) = 0$, where $u(r, z) = u_r(r, z), w(r, z) = u_z(r, z)$ are displacements, $p(r, z)$ is pore pressure, $\sigma_z(r, z), \tau_{rz}(r, z)$ are normal and tangential stress. The upper edge of the cylinder $z = 0$ is loaded $\sigma_z^-(r, 0) = -L(r), \tau_{rz}^-(r, 0) = T(r), p_-(r, 0) = P(r)$, where $L(r), T(r), P(r)$ are known functions. The bottom edge $z = h$ is fixed and permeable $u_+(r, h) = 0, w_+(r, h) = 0, p_+(r, h) = 0$. The displacements and pore pressure in each part Ω_-, Ω_0 and Ω_+ satisfy equations [2].

The asymptotic method [3] made it possible to derive imperfect transmission conditions between the main parts Ω_- and Ω_+ regarding the orders of poroelastic constants of the interlayer G_0 - shear modulus, k_0 - permeability coefficient, S_{p0} - storativity of the pore space, since two other constants α_0 - Biot's constant and κ_0 - Muskhelishvili's constant, always take values of the same order, in fixed segments. It allowed to derive 8 basic and 10 auxiliary transmission conditions. The basic transmission conditions have the following form:

1. G_0, k_0, S_{p0} are of the same order as corresponding constants of the main parts $G_\pm, k_\pm, S_{p\pm}$. In this case displacements, stress, pore pressure and normal flux are continuous through the interface;

2. $G_0 \ll G_\pm$. In this case there are jumps of the displacements, all other functions are continuous
$$u_0^+\big|_{z=d+0} - u_0^-\big|_{z=d-0} = \tfrac{h_0}{G_0}\tau_{rz,0}^-\big|_{z=d-0},$$
$$w_0^+\big|_{z=d+0} - w_0^-\big|_{z=d-0} = \tfrac{h_0}{G_0}\tfrac{\kappa_0-1}{\kappa_0+1}\sigma_{z,0}^-\big|_{z=d-0}$$

3. $k_0 \ll k_\pm$. In this case there is a jump of the pore pressure, all other functions are continuous
$$p_0^+\big|_{z=d+0} - p_0^-\big|_{z=d-0} = \frac{h_0}{k_0}k_-\frac{\partial p_0^-}{\partial z}\bigg|_{z=d-0}$$

4. $G_0 \gg G_\pm$. In this case there are jumps of the stress, all other functions are continuous
$$\sigma_{z,0}^+\big|_{z=d+0} - \sigma_{z,0}^-\big|_{z=d-0} = -G_0h_0\tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\tfrac{\partial w_0^-}{\partial r}\right)\Big|_{z=d-0},$$
$$\tau_{rz,0}^+\big|_{z=d+0} - \tau_{rz,0}^-\big|_{z=d-0} = G_0h_0\tfrac{\kappa_0+1}{\kappa_0-1}\left[\tfrac{1}{r^2}u_0^- - \tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\tfrac{\partial u_0^-}{\partial r}\right)\right]\Big|_{z=d-0}$$

5. $k_0 \gg k_\pm$. In this case there is a jump of the normal flux, all other functions are continuous
$$k_+\frac{\partial p_0^+}{\partial z}\bigg|_{z=d+0} - k_-\frac{\partial p_0^-}{\partial z}\bigg|_{z=d-0} = -k_0h_0\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_0^-}{\partial r}\right)\bigg|_{z=d-0}$$

6. $S_{p0} \gg S_{p\pm}$. In this case there is a jump of the normal flux, all other functions are continuous
$$k_+ \frac{\partial p_0^+}{\partial z}\bigg|_{z=d+0} - k_- \frac{\partial p_0^-}{\partial z}\bigg|_{z=d-0} = h_0 S_{p0} s\, p_0^-\big|_{z=d-0}$$

7. $G_0 \ll G_\pm, k_0 \ll k_\pm$. In this case there are jumps of the displacements, normal stress, pore pressure and normal flux, tangential stress functions are continuous
$$\begin{array}{l}
w_0^+\big|_{z=d+0} - w_0^-\big|_{z=d-0} = \left[\sigma_{z,0}^- \frac{(\kappa_0-1)\varpi}{G\alpha_0(\kappa_0+1)} + k_- \frac{\partial p_0^-}{\partial z}\frac{\vartheta}{\alpha_0 s}\right]\Big|_{z=d-0},\\
u_0^+\big|_{z=d+0} - u_0^-\big|_{z=d-0} = \frac{h_0}{G_0}\tau_{rz,0}^-\big|_{z=d-0},\\
p_0^+\big|_{z=d+0} - p_0^-\big|_{z=d-0} = \left[\sigma_{z,0}^- \frac{\vartheta}{\alpha_0} + k_- \frac{\partial p_0^-}{\partial z}\frac{\varpi}{\alpha_0 k}\right]\Big|_{z=d-0},\\
\sigma_{z,0}^+\big|_{z=d+0} - \sigma_{z,0}^-\big|_{z=d-0} = \left[\sigma_{z,0}^- \vartheta + k_- \frac{\partial p_0^-}{\partial z}\frac{G_0\varpi}{k_0 G}\right]\Big|_{z=d-0},\\
k_+ \frac{\partial p_0^+}{\partial z}\Big|_{z=d+0} - k_- \frac{\partial p_0^-}{\partial z}\Big|_{z=d-0} = \left[\sigma_{z,0}^- \frac{s(\kappa_0-1)\varpi}{G(\kappa_0+1)} + k_- \frac{\partial p_0^-}{\partial z}\vartheta\right]\Big|_{z=d-0},
\end{array}$$
where $\vartheta = 2\sinh^2\left(\frac{\alpha_0 W h}{2}\right), \varpi = \sinh\left(\alpha_0 W h\right)/W, W = \sqrt{\frac{(\kappa_0-1)}{Gk(\kappa_0+1)}}$.

8. $G_0 \ll G_\pm, k_0 \ll k_\pm, S_{p0} \gg S_{p\pm}$. In this case there are jumps of the displacements, and all other functions are continuous
$$\begin{array}{l}
u_0^+\big|_{z=d+0} - u_0^-\big|_{z=d-0} = \frac{h_0}{G_0}\tau_{rz,0}^-\big|_{z=d-0},\\
w_0^+\big|_{z=d+0} - w_0^-\big|_{z=d-0} = \frac{h_0}{G_0}\frac{S_p(\kappa_0-1)}{(\kappa_0+1)}\sigma_{z,0}^-\big|_{z=d-0}
\end{array}$$

Ten auxiliary transmission conditions coincide with the combination of the presented conditions.

The original problem with one of the listed transmission conditions is reduced to the one-dimensional boundary value problem using the Hankel integral transform applied regarding variable r. The derived problem is formulated as a vector boundary value problem, and its solution is constructed with the help of matrix differential calculation apparatus [4]. The displacements, stress, and pore pressure are found for each of the main parts Ω_- and Ω_+. The numerical study was conducted for a solid cylinder with all indicated transmission conditions under different external loadings.

References

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