Asymptotic properties of solutions to hight order differential equations with nonlinearities close to regularly varying

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Consider the following equation

$$y^{(n)} = \alpha_0 p(t) f(y, y', \dots, y^{(n-1)}) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $a < \omega \le +\infty$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$, $i = \overline{0, n-1}$, are continuous, regularly varying functions of order σ_i as $y^{(i)} \to Y_i$, $i = 0, \dots, n-1$, $Y_i \in \{0, \pm\infty\}$, $\sigma_i \in \mathbb{R}$, and $\sigma = \sum_{i=0}^{n-1} \sigma_i \ne 1$, Δ_{Y_i} is of the form either $[y_i^0, Y_i[$ or $]Y_i, y_i^0]$, $f : [a, \omega[\times \Delta_{Y_0} \times \Delta_{Y_1} \times \dots \times \Delta_{Y_{n-1}} \to]0, +\infty[$ is a continuously differentiable function such that

$$\lim_{\substack{v_k \to Y_k \\ v_k \in \Delta_{Y_k}}} \frac{v_k \cdot \frac{\partial f}{\partial v_k}(v_0, v_1, \dots, v_{n-1})}{f(v_0, v_1, \dots, v_{n-1})} = 0 \text{ uniformly with respect to } v_j \in \Delta_{Y_j} \tag{2}$$

for $j, k \in \{0, 1, \dots, n-1\}$, $j \ne k$.

It follows from (2) that the function $f(y, y', \dots, y^{(n-1)})$ is close to a regularly varying functions [1]. As examples of such functions one can consider $|y_0|^{\gamma_0} |y_1|^{\gamma_1} \dots |y_n|^{\gamma_n} \times \exp(\ln^\mu |y_0 y_1 \dots y_n|)$, $|y_0|^{\gamma_0} |y_1|^{\gamma_1} \dots |y_n|^{\gamma_n} \ln^\mu |y_0 y_1 \dots y_n| \ln |y_0 y_1 \dots y_n|$ and so forth. Therefore, class of differential equations (1) encompasses a class of high order essentially nonlinear equations. Partial cases of such equations are used for studying high order differential equations, which are used for modeling of complex physical processes. One of partial cases of (1) was studied in [2].

Definition 1. Let $-\infty \le \lambda_0 \le +\infty$. A function $y : [t_0, \omega[\to \mathbb{R}$ $(t_0 \in [a, \omega[)$, which is n times continuously differentiable, is called a $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solution of the differential equation (1) if it satisfies the conditions

$$y^{(n)}(t) \ne 0, \qquad y^{(j-1)}(t) \in \Delta_{Y_{j-1}}, \quad (j = 1, \dots, n), \quad t \in [t_0, \omega[,$$

$$\lim_{t \uparrow \omega} y^{(j-1)}(t) = Y_{j-1}, \quad (j = 1, \dots, n), \qquad \lim_{t \uparrow \omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}(t)\, y^{(n-2)}(t)} = \lambda_0,$$

and

$$y^{(n)}(t) = \alpha_0 p(t) f\left(y(t), y'(t), \dots, y^{(n-1)}(t)\right) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}(t)), \quad t \in [t_0, \omega[.$$

Necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions were established in the non-singular case when $\lambda_0 \in \mathbb{R} \setminus \{0, 1, \dots, n-2, 1\}$. Moreover, asymptotic representations of such solutions and their derivatives up to order $n-1$ inclusive as $t \uparrow \omega$ were obtained, and the question of the number of solutions with the derived asymptotic representations was resolved.

Let's introduce the following notations:

$$\pi_\omega(t) = \begin{cases} t, & \textit{if} \quad \omega = +\infty, \\ t-\omega, & \textit{if} \quad \omega < +\infty, \end{cases} \qquad I(t) = \int_{A_\omega}^{t} p(\tau)|\pi_\omega(\tau)|^{1-\sigma} d\tau,$$

$$A_\omega = \begin{cases} a, & \textit{if} \quad \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{1-\sigma} d\tau = +\infty, \\ \omega, & \textit{if} \quad \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{1-\sigma} d\tau < +\infty, \end{cases} \qquad \beta = \begin{cases} 1 & \textit{if} \ \omega = +\infty, \\ -1 & \textit{if} \ \omega < +\infty, \end{cases}$$

$$\nu_i = \operatorname{sign} Y_i, \quad a_{0i} = (n-i)\lambda_0 - (n-i-1), \quad C = \prod_{i=0}^{n-1} \left| \frac{(\lambda_0 - 1)^{n-i-1}}{\prod_{j=i+1}^{n-1} a_{0j}} \right|^{\sigma_i}, \quad (i = \overline{0, n-1}).$$

Then the following theorem holds:

Theorem 1. *Let* $\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \ldots, \frac{n-2}{n-1}\right\}$ *and* $1 - \sigma \neq 0$. *Then for the existence of a* $P_\omega(Y_0, Y_1, \ldots, Y_{n-1}, \lambda_0)$*-solution to the differential equation* (1) *the conditions*

$$\nu_{j-1}\nu_j < 0 \quad \textit{if} \quad Y_{j-1} = 0, \quad \nu_{j-1}\nu_j > 0 \quad \textit{if} \quad Y_{j-1} = \pm\infty,$$
$$\alpha_0\nu_{n-1} < 0 \quad \textit{if} \quad Y_{n-1} = 0, \qquad \alpha_0\nu_{n-1} > 0 \quad \textit{if} \quad Y_{n-1} = \pm\infty,$$

are necessary, as well as

$$\nu_{j-1}\sigma_j a_{0j}(\lambda_0 - 1)\pi_\omega(t) > 0 \quad (j = 1, \ldots, n), \qquad \alpha_0\nu_{n-1}(\lambda_0 - 1)\pi_\omega(t) > 0,$$
$$\alpha_0\nu_{n-1}^{\sigma-1} I(t) > 0 \quad \textit{for } t \in [a, \omega[, \qquad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) I'(t)}{I(t)} = \frac{1-\sigma}{\lambda_0 - 1},$$

and, if the algebraic equation with respect to μ

$$\frac{\beta}{\lambda_0 - 1} \sum_{i=1}^{n} \sigma_{i-1} \prod_{k=i}^{n-1} \beta \frac{a_{0k}}{\lambda_0 - 1} \prod_{k=1}^{i-1} \left(\beta \frac{a_{0k}}{\lambda_0 - 1} + \mu \right) = \left(\mu + \frac{\beta}{\lambda_0 - 1} \right) \prod_{k=1}^{n-1} \left(\beta \frac{a_{0k}}{\lambda_0 - 1} + \mu \right) \tag{3}$$

has no roots with zero real part, they are also sufficient.

Moreover, for each such solution, as $t \uparrow \omega$, *the asymptotic relations hold*

$$y^{(i)}(t) = \frac{[(\lambda_0 - 1)\pi_\omega(t)]^{n-i-1}}{\prod_{j=i+1}^{n-1} a_{0j}} y^{(n-1)}(t)[1 + o(1)], \quad (i = \overline{0, n-1}),$$

where $y^{(n-1)}(t)$ *is determined by the asymptotic relation*

$$\frac{|y^{(n-1)}(t)|^{1-\sigma}}{f\left(y(t), \ldots, y^{(n-1)}(t)\right) \prod_{i=0}^{n-1} \Theta_i(y^{(i)}(t))} = \nu_{n-1}\alpha_0 C(1-\sigma) I(t)[1 + o(1)]$$

in which $\Theta_i(y^{(i)}(t))$ *is a slowly varying component of the function* φ_i $(i = 0, \ldots, n-1)$.

There exists an m*-parametric family of solutions with these asymptotics in the case when among the roots of the algebraic equation* (3) *there are* m *roots (counting multiplicities), whose real parts have sign opposite to that of* $(\lambda_0 - 1)\pi_\omega(t)$.

[1] Seneta E., Regularly varying functions, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1976, 116 pp.

[2] Drozhzhyna A. V., Asymptotic behavior of solutions of nonlinear nonautonomous ordinary differential equations of the n-th order, PhD thesis, Odesa I. I. Mechnikov National University, Odesa, 2020.