MFS-based numerical scheme with Laguerre time semi-discretization for 2D wave processes in channels

Y. Yanchynskyi[1]

[1]Faculty of Applied Mathematics and Informatics, Ivan Franko National University of Lviv, Lviv, Ukraine

yurii.yanchynskyi@lnu.edu.ua

This paper considers the numerical solution of linear wave processes in two-dimensional models of water channels for both transverse and longitudinal sections. Let D be a section of a water channel filled with an inviscid incompressible liquid having a free surface. Denote by Γ_1 the free water boundary, by Γ_2 the wetted channel boundary and by Ω the water-filled domain bounded by those boundaries $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We seek the function $u : \Gamma_1 \times [0, T) \to \mathbb{R}$, which satisfies the evolution operator equation and the initial conditions

$$\frac{\partial^2 u}{\partial t^2} + Au = f \quad \text{on } \Gamma_1 \times (0, T], \tag{1}$$

$$u\Big|_{t=0} = w_1, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = w_2 \quad \text{on } \Gamma_1, \tag{2}$$

where w_1, w_2 and f are given sufficiently smooth functions on Γ_1, f describes the force field which acts on the moving fluid. The operator A is defined as $A\psi = \frac{\partial v}{\partial \nu}$ on $\Gamma_1 \times (0, T]$, where v is the solution of the corresponding mixed Dirichlet–Neumann boundary value problem

$$\Delta v = 0 \quad \text{in } \Omega, \tag{3}$$

$$v = \psi \quad \text{on } \Gamma_1 \qquad \text{and} \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_2. \tag{4}$$

Here ν denotes the outward unit normal vector to the boundaries Γ_1 and Γ_2 and we use the notation $x = (x_1, x_2)$. The well-posedness of the hyperbolic evolution problem is discussed in [4].

In this study, we consider a widely used two-step approach. First, we apply semi-discretization with respect to the time variable by applying the Laguerre transform [2] to obtain a sequence of elliptic boundary value problems with the recurrent operator equations. This step is summarized in the following theorem

Theorem 1. *The solution of the problem (1)-(2) can be presented in the form*

$$u(x, t) = \kappa \sum_{n=0}^{\infty} L_n(\kappa t)\, u_n(x), \qquad x \in \Gamma_1,\ t \in (0, T], \tag{5}$$

where L_n are the Laguerre polynomials of order n [1], $\kappa > 0$ is a given constant and functions u_n satisfy the sequence of recurrent operator equations

$$(A + \beta_0 I)u_n = F_n - \sum_{m=0}^{n-1} \beta_{n-m} u_m \quad \text{on } \Gamma_1, \quad n = 0, 1, \ldots \tag{6}$$

Here I is the identity operator, $\beta_n = \kappa^2(n + 1)$, $F_n(x) = f_n(x) + w_2(x) + \kappa(n + 1)w_1(x)$, $x \in \Gamma_1$ and f_n are Fourier–Laguerre coefficients of the function f for $n = 0, 1, \ldots$

At the next step we numerically solve the obtained recurrent operator equations using the method of fundamental solutions (MFS) while also taking into account the specifics of longitudinal and cross-sections of the channel. Specifically, in the case of cross-section the numerical solutions of the boundary value problems (3)-(4) is sought in form

$$v_{n,M}(x) = \sum_{k=1}^{M} \alpha_{n,k} \Phi(x, y_k), \qquad x \in \Omega,\ n = 0, \ldots, N, \tag{7}$$

where $M \in \mathbb{N}$, $y_k \notin \overline{\Omega}$, $k = 1, \ldots, M$ are selected source points, $\Phi(x, y)$ is a fundamental solution of the Laplace equation in $\mathbb{R}^2$ and $\alpha_{n,k} \in \mathbb{R}$, $k = 1, \ldots, M$ are coefficients to be determined by the collocation method on boundaries Γ_1, Γ_2. Next we describe the rules for selecting the collocation and source points using boundary parametrizations. Having chosen the required points we can find the unknown coefficients $\alpha_{n,k}$ for $n = 0, 1, \ldots, N$. By substituting the approximation (7) into (6) and (4), we obtain the sequence of linear equations

$$\begin{cases} \sum_{k=1}^{M} \alpha_{n,k} \left[\beta_0 \Phi(\hat{x}_i, y_k) + \Psi(\hat{x}_i, y_k)\right] = B_{n,i}, & i = 1, \ldots, \hat{M}, \\ \sum_{k=1}^{M} \alpha_{n,k} \Psi(\tilde{x}_i, y_k) = 0, & i = 1, \ldots, \tilde{M}, \end{cases} \tag{8}$$

where $\Psi(x, y) = \frac{\partial \Phi(x,y)}{\partial \nu(x)}$, $\hat{M}$ and $\tilde{M}$ are the numbers of collocation points $\hat{x}_i$, $\tilde{x}_i$ on boundaries Γ_1, Γ_2 respectfully,

$$B_{n,i} = F_n(\hat{x}_i) - \sum_{m=\ddot{m}}^{n-1} \beta_{n-m} v_{m,M}(\hat{x}_i), \quad i = 1, \ldots, \hat{M} \tag{9}$$

and $v_{m,M}$ are obtained using the found coefficients $\alpha_{m,k}$ from (8) for $m = 0, \ldots, n-1$. We also describe the approximation error of applied MFS scheme.

Finally, we present the results of numerical examples for each type of channel section and conclusions that demonstrate the efficiency of implemented numerical approach.

The author is grateful to the supervisor and colleagues from the Department of Computational Mathematics at Ivan Franko National University of Lviv for valuable comments and discussions related to the numerical solution of wave process problems.

[1] Abramowitz M., Stegun I.A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, Washington, D.C., 1972, 775–779.

[2] Chapko R., Datsiv G., The numerical solution of the axially symmetric linear sloshing problem by the boundary integral equation method, *Journal of Integral Equations and Applications* **20** (2008), no. 4, 409–436.

[3] Chapko R., Kress R., On the numerical solution of initial boundary value problems by the Laguerre transformation and boundary integral equations, in *Integral and Integrodifferential Equations: Theory, Methods and Applications*, vol. 2, Gordon and Breach Science Publishers, Amsterdam, 2000, 55–69.

[4] Garipov R.M., On the linear theory of gravity waves: the theorem of existence and uniqueness, *Arch. Rat. Mech. Anal.* **24** (1967), 352–362.