Classification of symmetry properties of a class of systems of diffusion equations

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In this work, we study the symmetry properties of a subclass of nonlinear diffusion systems of the form

$$U_t = \partial_x \left[F(U) U_x \right],$$

where $U \in \mathbb{R}^2$ and $F(U)$ is a diffusion matrix. Specifically, we consider systems in which the elements of the diffusion matrix exhibit a power-law dependence on u^1, corresponding to the so-called third class of diffusion systems (see [1] for details). This subclass consists of systems of the form

$$\begin{aligned} u^1_t &= \partial_x \left[(u^1)^n \varphi^{11} u^1_x + (u^1)^{n+1} \varphi^{12} u^2_x \right], \\ u^2_t &= \partial_x \left[(u^1)^{n-1} \varphi^{21} u^1_x + (u^1)^n \varphi^{22} u^2_x \right], \end{aligned} \tag{1}$$

where $\varphi^{ab} = \varphi^{ab}(u^2)$ are arbitrary smooth functions and n is a parameter. The symmetry analysis is conducted using the classical Lie method, enabling us to determine the vector fields that leave the system invariant. We derive the determining equations for the coefficients of the infinitesimal operator and establish their general structure. We show that any system within this considered class admits a principal (kernel) invariance algebra generated by time and space translations, alongside a scaling operator. Further analysis, based on splitting the determining equations, yields constraints on the functional forms of $\varphi^{ab}(u^2)$. The classification problem ultimately reduces to investigating several non-equivalent cases corresponding to different functional forms of the diffusion coefficients. In particular, we identify a nontrivial case where the system can be written as

$$\begin{aligned} u^1_t &= \partial_x \left[(u^1)^n (u^2)^m \left(\lambda_{11} u^1_x + \lambda_{12} \frac{u^1}{u^2} u^2_x \right) \right], \\ u^2_t &= \partial_x \left[(u^1)^n (u^2)^m \left(\lambda_{21} \frac{u^2}{u^1} u^1_x + \lambda_{22} u^2_x \right) \right], \end{aligned}$$

where n and m are arbitrary parameters, and λ_{11}, λ_{12}, λ_{21}, and λ_{22} are constants of integration. For this specific subclass, we obtain the kernel of the maximal Lie invariance algebras and derive the parameter values for which these nonlinear diffusion systems admit extensions of the Lie invariance algebra. Finally, we provide a complete symmetry classification for the third class of nonlinear diffusion systems (1). These results can be applied to construct exact solutions, perform reduction procedures, and further analyze models arising in nonlinear diffusion processes.

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