

ON THE RELATIONSHIP BETWEEN RESOLVENTS AND FRACTIONAL POWERS OF GENERATORS OF OPERATOR SEMIGROUPS

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This paper studies integral and operator relations associated with C_0 -semigroups of linear operators in a Banach space. Such semigroups play an important role in functional analysis, the theory of differential equations, and spectral theory of operators [1].

Let X be a Banach space and let $E(X)$ be the Banach algebra of bounded linear operators on X . Consider a one-parameter family of operators $\{T(t), t \geq 0\}$ forming a C_0 -semigroup. Denote by A its infinitesimal generator and by $R(\lambda, A)$ the resolvent of the operator A .

The main result of the paper is the derivation of an integral representation that connects products of resolvents with fractional powers of the generator (see, e.g., [2, 3]).

Theorem 1. *Assume that the resolvent $R(\lambda, A)$ exists for $\operatorname{Re} \lambda > 0$ and satisfies the condition*

$$\sup_{\operatorname{Re} \lambda > 0} |\operatorname{Re} \lambda| \|R(\lambda, A)\| < \infty.$$

Then for $m - 1 < \gamma < m \leq n$ and $f \in D(A^m)$ the following equality holds:

$$\int_0^\infty t^{\gamma-1} R\left(\frac{t}{n}, A\right) \cdots R\left(\frac{t}{n-m+1}, A\right) A^m f dt = -\frac{\pi W(\gamma, m, n)}{\sin \pi \gamma} (-A)^\gamma f,$$

where

$$W(\gamma, m, n) = \binom{n}{m} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} (m-k)(n-k)^{\gamma-1}.$$

To prove this result, auxiliary combinatorial and operator identities are obtained. In particular, the identity

$$\binom{n}{m} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} \frac{m-k}{n-k} x^{n-k} + (-1)^m = (x-1)^m \sum_{k=m}^n \binom{k-1}{m-1} x^{n-k}$$

is used, as well as operator equalities for products of resolvents based on Hilbert's identity

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

The obtained relations make it possible to connect different representations of fractional powers of operators, in particular in the sense of Balakrishnan [2], with integral expressions involving semigroups of operators.

As an illustration, consider the simplest semigroup

$$T(u)f = e^{-u}f.$$

In this case $Af = -f$, and

$$R(\lambda, A)f = \frac{f}{1 + \lambda}.$$

The obtained relations lead to the integral formula

$$n(n-1)\cdots(n-m+1)\int_0^\infty \frac{t^{\gamma-1} dt}{(t+n)(t+n-1)\cdots(t+n-m+1)} = -\frac{\pi W(\gamma, m, n)}{\sin \pi \gamma}.$$

Thus, the paper establishes relations connecting resolvents of an operator, semigroups, and fractional powers of the generator, which are also closely related to Bernstein functions [4]. These results are of interest for further development of the theory of operator semigroups and its applications.

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