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N-Soliton Solutions to the Korteweg-de Vries Equation

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The Korteweg-de Vries equation is one of the most important nonlinear evolution equations in mathematical sciences. It is fascinating example of propagation of weakly dispersive and weakly nonlinear waves. The Korteweg-de Vries equation of the next simplified form

$$u_t = 6uu_x - u_{xxx} \tag{1}$$

is partial differential equation of the third order that describes certain types of wave phenomena in shallow water. The solution to this equation describes one mysterious wave, discovered by John Scott Russell in 1834, and is named soliton. Nowadays it is hard to understand for mathematicians not specialized in fluid mechanics, that a subject as this could raise indeed wide world-spread interest. However, this was not the case in the XIX century when the study of water waves was of vital interest for applications in naval architecture. Notably in England and France much research was spent on water waves of several kinds, in England by, among others, Airy, Stokes, McCowan, Lord Rayleigh, and Lamb, in France — by Lagrange, Clapeyron, Bazin, St. Venant, and Boussinesq. Soliton waves were later found in models of plasma, solid-state physics, biological systems, optic systems.

In the middle of 1960's the Korteweg-de Vries equation was found out to be closely related to the Schrödinger equation. Due to this fact, this equation is interpreted as sort of condition for compatibility of an abstract system of two auxiliary linear equations, and the solution of this equation can be found with the help of the Darboux transformation apparatus.

In the given paper we consider mathematical application of the Darboux transformation in order to obtain solutions of the Korteweg-de Vries equation in explicit form. With the help of this transform explicit form of N-soliton, $N \geq 1$, has been obtained only for the simplest case $N = 1$, spread in soliton-dedicated literature. After analysis of number of references, both classical old and modern, we have found out that explicit form of multisoliton solutions is unknown. Our conclusions were based on the formula

$$u[N] = u - 2\big(lnW[\Psi_1, \Psi_2, \ldots, \Psi_N]\big)_{xx}$$

from the Darboux transformation apparatus and aimed to find general formula for N-soliton, $N \geq 1$, solutions of (1), expressed in terms of eigenfunctions Ψ_N, $N \geq 1$, of auxiliary spectral problem for one-dimensional Schrödinger operator.

We have put for $\alpha \neq 0$

$$u = 0,\ \Psi_1 = ch\big[\alpha(x - x_1) - 4\alpha^3 t\big],\ \Psi_2 = sh\big[\alpha(x - x_2) - 4\alpha^3 t\big],\ \lambda = -\alpha^2,$$

and have obtained two-soliton solution $u[2]$ to (1) of the form

$$u[2] = -2\big(lnW[\Psi_1, \Psi_2]\big)_{xx} = -2\frac{W_{xx}[\Psi_1, \Psi_2]W[\Psi_1, \Psi_2] - (W_x[\Psi_1, \Psi_2])^2}{W^2[\Psi_1, \Psi_2]} = 0,$$

where

$$W[\Psi_1, \Psi_2] = \begin{vmatrix} \Psi_1 & \Psi_2 \\ \Psi_{1x} & \Psi_{2x} \end{vmatrix} = \alpha\big(ch^2\big[\alpha(x - x_1) - 4\alpha^3 t\big] - sh^2\big[\alpha(x - x_1) - 4\alpha^3 t\big]\big) = \alpha,$$

$$W_x[\Psi_1,\Psi_2]=\begin{vmatrix}\Psi_1 & \Psi_2\\ \Psi_{1xx} & \Psi_{2xx}\end{vmatrix}=\alpha^2 ch\big[\alpha(x-x_1)-4\alpha^3t\big]sh\big[\alpha(x-x_2)-4\alpha^3t\big]-$$
$$-\alpha^2 ch\big[\alpha(x-x_1)-4\alpha^3t\big]sh\big[\alpha(x-x_2)-4\alpha^3t\big]=0,$$
$$W_{xx}[\Psi_1,\Psi_2]=\begin{vmatrix}\Psi_{1x} & \Psi_{2x}\\ \Psi_{1xx} & \Psi_{2xx}\end{vmatrix}+\begin{vmatrix}\Psi_1 & \Psi_2\\ \Psi_{1xxx} & \Psi_{2xxx}\end{vmatrix}=\alpha^3 sh\big[\alpha(x-x_1)-4\alpha^3t\big]sh\big[\alpha(x-x_2)-4\alpha^3t\big]-$$
$$-\alpha^3 ch\big[\alpha(x-x_1)-4\alpha^3t\big]ch\big[\alpha(x-x_2)-4\alpha^3t\big]+\alpha^3 ch\big[\alpha(x-x_1)-4\alpha^3t\big]ch\big[\alpha(x-x_2)-4\alpha^3t\big]-$$
$$-\alpha^3 sh\big[\alpha(x-x_1)-4\alpha^3t\big]sh\big[\alpha(x-x_2)-4\alpha^3t\big]=0.$$

We have put for $\alpha\neq 0$

$u=0,\ \Psi_1=ch\big[\alpha(x-x_1)-4\alpha^3t\big],\ \Psi_2=sh\big[\alpha(x-x_2)-4\alpha^3t\big],\ \Psi_3=ch\big[\alpha(x-x_3)-4\alpha^3t\big],\ \lambda=-\alpha^2,$ and have obtained three-soliton solution $u[3]$ to (1) of the form

$$u[3]=u-2\big(lnW[\Psi_1,\Psi_2,\Psi_3]\big)_{xx}=-2\big(lnW[\Psi_1,\Psi_2,\Psi_3]\big)_{xx}=$$
$$=-2\frac{W_{xx}[\Psi_1,\Psi_2,\Psi_3]W[\Psi_1,\Psi_2,\Psi_3]-(W_x[\Psi_1,\Psi_2,\Psi_3])^2}{W^2[\Psi_1,\Psi_2,\Psi_3]}=$$
$$=-\frac{2\alpha^4\big(\alpha^2-\alpha^2\big)^2ch^2\big[\alpha(x_2-x_1)\big]}{\alpha^2\big(\alpha^2-\alpha^2\big)^2ch^2\big[\alpha(x-x_3)-4\alpha^3t\big]ch^2\big[\alpha(x_2-x_1)\big]}=-\frac{2\alpha^2}{ch^2\big[\alpha(x-x_3)-4\alpha^3t\big]},$$

where

$$W[\Psi_1,\Psi_2,\Psi_3]=\begin{vmatrix}\Psi_1 & \Psi_2 & \Psi_3\\ \Psi_{1x} & \Psi_{2x} & \Psi_{3x}\\ \Psi_{1xx} & \Psi_{2xx} & \Psi_{3xx}\end{vmatrix}=\alpha\big(\alpha^2-\alpha^2\big)ch\big[\alpha(x-x_3)-4\alpha^3t\big]ch\big[\alpha(x_2-x_1)\big],$$
$$W_x[\Psi_1,\Psi_2,\Psi_3]=\begin{vmatrix}\Psi_1 & \Psi_2 & \Psi_3\\ \Psi_{1x} & \Psi_{2x} & \Psi_{3x}\\ \Psi_{1xxx} & \Psi_{2xxx} & \Psi_{3xxx}\end{vmatrix}=\alpha^2\big(\alpha^2-\alpha^2\big)sh\big[\alpha(x-x_3)-4\alpha^3t\big]ch\big[\alpha(x_2-x_1)\big],$$
$$W_{xx}[\Psi_1,\Psi_2,\Psi_3]=\begin{vmatrix}\Psi_1 & \Psi_2 & \Psi_3\\ \Psi_{1xx} & \Psi_{2xx} & \Psi_{3xx}\\ \Psi_{1xxx} & \Psi_{2xxx} & \Psi_{3xxx}\end{vmatrix}+\begin{vmatrix}\Psi_1 & \Psi_2 & \Psi_3\\ \Psi_{1x} & \Psi_{2x} & \Psi_{3x}\\ \Psi_{1xxxx} & \Psi_{2xxxx} & \Psi_{3xxxx}\end{vmatrix}=$$
$$=\alpha^3\big(\alpha^2-\alpha^2\big)ch\big[\alpha(x-x_3)-4\alpha^3t\big]ch\big[\alpha(x_2-x_1)\big].$$

Continuing in a similar way, we obtain

$$u[2N-1]=-\frac{2\alpha^2}{ch^2\big[\alpha(x-x_{2N-1})-4\alpha^3t\big]},\ N\geq 1,$$
$$u[2N]=0,\ N\geq 1.$$

Using the Darboux transformation, we are able to obtain all N-soliton, $N\geq 1$, solutions of the Korteweg-de Vries equation, that are identified with series of potentials of one-dimensional Schrödinger operator in so-called Sturm-Liouville problem. This result is supposed to become the first step in finding general formula for N-soliton solutions, expressed in terms of functions Ψ_N, $N\geq 1$, depending on different, not unique one, parameters. We also hope soliton lovers will be interested in this brief review and will use it as helpful reference in their research.

1. Ablowitz M. J., Segur H., Solitons and the inverse scattering transform, SIAM, Philadelphia, 1981, 425 pp.
2. Matveev V. B., Salle M. A., Darboux transformations and solitons, Springer, Berlin, 1991, 120 pp.