FORWARD AND INVERSE CRUM TRANSFORMATION OF THE LAGUERRE OPERATORS

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We study the classical and non-classical Laguerre operators ℓ_α on $L^2\big(0,+\infty\big)$ in terms of the Crum transformation, which can be defined by

$$\ell_\alpha = -\frac{d^2}{dx^2} + \frac{\alpha^2 - \frac{1}{4}}{x^2} + x^2.$$

Note, if $\alpha > -1$, then ℓ_α is called a classical Laguerre operator. In this case, ℓ_α is considered in the Hilbert space $L^2(\mathbb{R}_+, \omega_\alpha)$, where $\omega_\alpha(x) = x^\alpha e^{-x}$ is a weight function. If $\alpha < -1$ and $\alpha \notin \mathbb{Z}_-$, ℓ_α is called a non-classical Laguerre operator the Pontryagin space $\Pi(\alpha)$ with the indefinite metric

$$\langle f, g\rangle_\alpha = \int\limits_0^\infty x^\alpha \left(e^{-x} f\overline{g} - \sum_{j=0}^{n-1} (e^{-x} f\overline{g})^{(j)}(0)\frac{x^j}{j!} \right) dx.$$

As was known,

$$\ell_\alpha(\phi_n(x,\alpha)) = \lambda_n \phi_n(x,\alpha), \quad n \in \mathbb{Z}_+,$$

where λ_n and ϕ_n are eigenvalue and eigenfunction, respectively, which can be found by

$$\phi_n(x,\alpha) = e^{-\frac{x^2}{2}} x^{\alpha+\frac{1}{2}} L_n(x^2,\alpha) \quad \text{and} \quad \lambda_n = 2\alpha + 2 + 4n, \quad n \in \mathbb{Z}_+.$$

Moreover, ϕ_n is called a Laguerre function and L_n is called a Laguerre polynomial such that

$$L_n(x,\alpha) = \frac{1}{n!} e^x x^{-\alpha} \left(e^{-x} x^{n+\alpha}\right)^{(n)}, \quad n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \text{ and } \alpha \notin \mathbb{Z}_-.$$

We obtained the following results:

Proposition 1. *Let $\alpha \notin \mathbb{Z}_-$ and $\phi_0, \dots, \phi_{n-1}$ be the Laguerre functions. Then the Wronskian $W(\phi_0(x,\alpha), \dots, \phi_{n-1}(x,\alpha))$ can be calculated by*

$$W(\phi_0(x,\alpha), ..., \phi_{n-1}(x,\alpha)) = \prod_{k=0}^{n-1} (-2)^k \phi(x, \alpha + k).$$

Next, we study the direct Crum transformation of the Laguerre operator.

Theorem 1. *Let $\alpha \notin \mathbb{Z}_-$, let ℓ_α be a Laguerre operator, and let $\phi_0, \dots, \phi_{n-1}$ be the Laguerre functions. Then the Crum transformation of ℓ_α is the following operator*

$$\ell^C = -\frac{d^2}{dx^2} + q^C,$$

where the potential q^C is given by

$$q^C(x) = \frac{(\alpha+n)^2 - \frac{1}{4}}{x^2} + x^2 + 2n.$$

Furthermore, ℓ^C can be rewritten in terms of Laguerre operators as

$$\ell^C = \ell_{\alpha+n} + 2n,$$

where is the Laguerre operator.

Our main goal is the inverse Crum transformation of the Laguerre operator and the relation between the classical and non-classical Laguerre operators.

Theorem 2. *Let ℓ_α be a Laguerre operator such that $n \in \mathbb{N}$ and $\alpha - n \notin \mathbb{Z}_-$, and let $\phi_0, \dots, \phi_{n-1}$ be the eigenfunctions of ℓ_α. Then the inverse Crum transformation of ℓ_α can be defined by*

$$\ell^{IC} = -\frac{d^2}{dx^2} + q^{IC},$$

where the potential q^{IC} can be calculated by

$$q^{IC}(x) = q_\alpha(x) - 2\frac{d^2}{dx^2} ln \left(\frac{1}{W(\phi_0(x,\alpha), ..., \phi_{n-1}(x,\alpha))} \right) + 2n + \frac{2n^2}{x^2}.$$

Furthermore, ℓ^{IC} is the Laguerre operator such that

$$\ell^{IC} = \ell_{\alpha-n}, \quad q^{IC}(x) = q_{\alpha-n}(x),$$

The point spectrum of the inverse Crum ℓ^{IC} can then be found by

$$\sigma_p(\ell^{IC}) = \sigma_p(\ell_\alpha) \cup \{\lambda_{-k}, ..., \lambda_{-1}\}, \quad \lambda_j = 2\alpha + 2 + 4j.$$

Corollary 1. *Let ℓ_α be a classical Laguerre operator such that $n \in \mathbb{N}$, $\alpha - n \notin \mathbb{Z}_-$ and $\alpha - n < -1$, and let $\phi_0, \dots, \phi_{n-1}$ be the eigenfunctions of ℓ_α. Then the inverse Crum transformation of ℓ_α is the non-classical Laguerre operator $\ell_{\alpha-n}$.*

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