Density convergence of spatial average of solution to a one dimensional stochastic wave equation

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Consider the following one-dimensional stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}, \qquad t \in \mathbb{R}^+, x \in \mathbb{R} \tag{1}$$

with initial condition $u(0,x) = 1$, $\frac{\partial}{\partial t}u(0,x) = 0$ (we assume this simple initial condition to focus our study on the stochastic part, other initial conditions can be discussed in the same way), where $\dot{W}$ is a Gaussian noise that is white in time and has a homogeneous spatial covariance described by the Riesz kernel. That is to say, the covariance of the centered Gaussian noise $\dot{W}$ is given by

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)|x-y|^{-\beta}$$

where $\beta \in (0,1)$ and δ_0 denotes the Dirac delta function at zero. This also means that $\dot{W}$ is white in time and fractional in space with Hurst parameter $H = 1 - \beta/2$. For the diffusion coefficient σ we introduce the following assumption:
(H) $\sigma : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function, so that σ', σ'' are bounded and $\sigma(x) \geq c > 0$ for all $x \in \mathbb{R}$.

We are interested in the central limit theorems for the density of the spatial averages of solutions. More specifically, for an $R > 0$ let the normalized spatial averages be defined by

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left(\int_{-R}^{R} [u(t,x) - 1] dx \right), \text{ where } \sigma_{R,t}^2 := \operatorname{Var}\left(\int_{-R}^{R} u(t,x) dx \right). \tag{2}$$

Our main result is

Theorem 1. *Let Assumption **(H)** be satisfied. Let $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the mild solution to the stochastic wave equation* (1) *with initial condition $u(0,x) = 1$, $\frac{\partial}{\partial t}u(0,x) = 0$. Let $F_{R,t}$ be defined as in* (2)*. Then, for all $R \geq 1$, $F_{R,t}$ has a density $f_{F_{R,t}}$, and there is a constant C_t, independent of R, such that*

$$\sup_{z \in \mathbb{R}} |f_{F_{R,t}}(z) - \phi(z)| \leq \begin{cases} C_t R^{-\beta/2}, & \beta \in (0, \frac{1}{2}), \\ C_t R^{-\beta/2} (\log R)^{1/2}, & \beta = \frac{1}{2}, \\ C_t R^{(\beta-1)/2}, & \beta \in (\frac{1}{2}, 1), \end{cases} \tag{3}$$

where ϕ is the density of a standard normal distribution on $\mathbb{R}$.

Results related to the stochastic heat equation have previously been established in [1,2]. In contrast to stochastic heat equations, for stochastic wave equation (1) the solution is no longer necessarily positive. This makes our task much more challenging.

The presentation of the results is based on authors' recent work in [3].

[1] Kuzgun S. and Nualart D., Convergence of densities of spatial averages of stochastic heat equation, *Stochastic Processes and Their Applications*, **151** (2022), 68-100.

[2] Kuzgun S. and Nualart D., Convergence of densities of spatial averages of the Parabolic Anderson model driven by colored noise, *Stochastics*, **96** (2024), 2, 968-984.

[3] Sun C. and Hu Y., Density convergence of spatial average of solution to a one dimensional stochastic wave equation, 2025, 23 pp., arXiv:2508.01872.