Convolution equations for the Euler copolynomial

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Copolynomials, introduced and studied in [1–3], are K-linear functionals on polynomial rings over a commutative ring K. They form an algebraic analogue of distributions, providing a natural language for the differential and convolution equations in algebraic settings. In the talk we consider one-variable copolynomials and characterize the Euler copolynomial by convolution equations.

Let K be a commutative ring with identity. A copolynomial in one variable over K is a K-linear mapping from the polynomial ring $K[x]$ to K. The set of all such mappings is denoted by $K[x]'$. For the value of $T \in K[x]'$ on $p \in K[x]$ we write (T, p). Let $\delta \in K[x]'$ be the delta-function, $(\delta, p) = p(0)$. For a polynomial $q \in K[x]$ and a copolynomial $T \in K[x]'$ we put $(qT, p) = (T, qp)$. We also use the derivative of a copolynomial,

$$(T', p) = -(T, p'), \qquad p \in K[x].$$

Following the convention of [2], the expression $p^{(k)}/k!$ is defined algebraically, so no division in K is required.

Following [1], we consider $K[x]'$ with the topology of pointwise convergence: $T_j \to T$ if, for every $p \in K[x]$, the values (T_j, p) are eventually equal to (T, p). A series of copolynomials converges if its partial sums converge in this topology. If $T \in K[x]'$, its Cauchy–Stieltjes transform is an element of $s^{-1}K[[s^{-1}]]$, and the transform and its inverse are given by

$$C(T)(s) = \sum_{k=0}^{\infty} (T, x^k) s^{-k-1}, \qquad C^{-1}\left(\sum_{k=0}^{\infty} b_k s^{-k-1}\right) = T, \quad (T, x^k) = b_k.$$

The multiplication of copolynomials is defined by

$$T_1 T_2 = C^{-1}\big(C(T_1)C(T_2)\big),$$

the convolution is defined by

$$(T_1 * T_2, p) = \sum_{k=0}^{\deg p} \left(T_1, \frac{p^{(k)}}{k!}\right) (T_2, x^k), \qquad p \in K[x].$$

For $a \in K$ we define the Euler copolynomial E_a by

$$E_a = \sum_{n=0}^{\infty} a^n n! \delta^{n+1}, \qquad \text{or equivalently} \qquad (E_a, x^k) = a^k k!, \quad k = 0, 1, 2, \ldots.$$

The series converges in the pointwise topology described above. If $K = \mathbb{R}$ and $a > 0$, then

$$(E_a, x^k) = \frac{1}{a}\int_0^{+\infty} x^k e^{-x/a}\, dx, \qquad k = 0, 1, 2, \dots \qquad .$$

This representation connects E_a with the classical Laplace transform of monomials. The following result gives convolution and differential characterizations of this copolynomial.

Theorem 1. *Let $T \in K[x]'$ and let $a \in K$. The following conditions are equivalent:*

(i) $T = E_a$;

(ii) $aT' + T = \delta$;

(iii) $a\,\delta\,(T * T) + \delta = T$;

(iv) $(T, 1) = 1$ *and* $a\,(T * T) = xT$.

In particular, for $a = 1$ the Euler copolynomial $E = E_1$ is characterized by the convolution equations

$$\delta\,(E * E) + \delta = E, \qquad \begin{cases} E * E = xE, \\ (E, 1) = 1. \end{cases}$$

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