Minimum loss strategy in the conflict game: the uniform defense

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We study a mathematical model of conflict between two opponents, A and B, that was originally introduced for a territory partitioned into three regions [1]. In the present work, we extend this model to an arbitrary number $n \geq 2$ of regions and investigate the problem of minimization of losses.

The conflict evolves in discrete time $t = 1, 2, \ldots$ The state at each step is described by total resource masses M^t, N^t and strategy vectors $\mathbf{p}^t, \mathbf{r}^t \in \mathbb{R}^n_+$ with

$$\sum_{i=1}^{n} p_i^t = \sum_{i=1}^{n} r_i^t = 1.$$

The regional masses are defined as

$$M_i^t = M^t p_i^t, \quad N_i^t = N^t r_i^t. \tag{1}$$

The total masses evolve according to

$$M^{t+1} = M^t(1 - \theta_{\mathbf{p}}^t) = M^{t=0} D_{\mathbf{p}}^t, \quad N^{t+1} = N^t(1 - \theta_{\mathbf{r}}^t) = N^{t=0} D_{\mathbf{r}}^t, \tag{2}$$

where $M^{t=0}$, $N^{t=0}$ are the initial masses, and

$$D_{\mathbf{p}}^t = \prod_{l=0}^{t}(1 - \theta_{\mathbf{p}}^l), \quad D_{\mathbf{r}}^t = \prod_{l=0}^{t}(1 - \theta_{\mathbf{r}}^l)$$

are the cumulative survival multipliers. The instantaneous loss rates are given by

$$\theta_{\mathbf{p}}^t = \sum_{i=1}^{n} \left(p_i^t\right)^2 r_i^t, \quad \theta_{\mathbf{r}}^t = \sum_{i=1}^{n} p_i^t \left(r_i^t\right)^2.$$

Simultaneously, the strategy vectors are updated by the stochastic conflict law (see [2])

$$p_i^{t+1} = \frac{p_i^t(1 - r_i^t)}{1 - \theta^t}, \quad r_i^{t+1} = \frac{r_i^t(1 - p_i^t)}{1 - \theta^t}, \quad \theta^t = \sum_{i=1}^{n} p_i^t r_i^t. \tag{3}$$

This n-dimensional system is a direct generalization of the modified Lotka–Volterra-type equations from [1]. The following theorem extends the convergence result of [1] to arbitrary n.

Theorem 1. *Every trajectory of the dynamical system, defined by the difference equations (1)–(3) for an arbitrary number of regions $n \geq 2$, converges to one of the fixed points of the set $\Gamma := \{M_i^\infty;\ N_i^\infty\} \subset \mathbb{R}^{2n}_+$, where*

$$M_i^\infty = \lim_{t\to\infty} M_i^t, \quad N_i^\infty = \lim_{t\to\infty} N_i^t, \quad i = 1, \ldots, n.$$

Moreover, $M_i^\infty > 0$ if and only if $\left(M_i^{t=0} N^{t=0}\right)/\left(N_i^{t=0} M^{t=0}\right) > 1$ (equivalently, $p_i^0 > r_i^0$). Similarly, $N_k^\infty > 0$ for $k \neq i$ if and only if $\left(N_k^{t=0} M^{t=0}\right)/\left(M_k^{t=0} N^{t=0}\right) > 1$ (equivalently, $r_k^0 > p_k^0$). In all other cases $M_j^\infty = N_j^\infty = 0$.

The limit values M_i^∞ and N_i^∞ admit the explicit representations $M_i^\infty = M^{t=0}\, D_{\mathbf{p}}^\infty\, p_i^\infty$ and $N_i^\infty = N^{t=0}\, D_{\mathbf{r}}^\infty\, r_i^\infty$, where

$$p_i^\infty = \lim_{t\to\infty} p_i^t, \quad r_i^\infty = \lim_{t\to\infty} r_i^t$$

are the stationary coordinates of the strategy vectors. The cumulative survival multipliers

$$D_{\mathbf{p}}^\infty = \prod_{t=0}^{\infty}(1-\theta_{\mathbf{p}}^t), \quad D_{\mathbf{r}}^\infty = \prod_{t=0}^{\infty}(1-\theta_{\mathbf{r}}^t)$$

are strictly positive whenever $\mathbf{p}^0 \neq \mathbf{r}^0$, and entirely determine the final state of the system.

One of the main results concerns the *uniform strategy* $\mathbf{u} = (1/n, \ldots, 1/n)$.

Theorem 2. *Let opponent B employ the uniform strategy $\mathbf{r}^0 = \mathbf{u}$. Then for any non-uniform strategy $\mathbf{p}^0 \neq \mathbf{u}$ of opponent A, the following inequality holds for all $t \geq 0$:*

$$\theta_{\mathbf{p}}^t > \theta_{\mathbf{r}}^t,$$

and consequently the relative accumulated losses satisfy

$$\frac{\Delta M^\infty}{M} > \frac{\Delta N^\infty}{N},$$

where $\Delta M^\infty = M - M^\infty$ and $\Delta N^\infty = N - N^\infty$.

Thus, the uniform defense perfectly dissipates any concentrated attack across the entire n-dimensional space. While the aggressor may achieve local dominance in certain regions, the uniform strategy guarantees that the total survival ratio $D_{\mathbf{r}}^\infty > D_{\mathbf{p}}^\infty$ always holds. This establishes the uniform distribution as a non-exploitable, globally optimal defensive configuration in the n-regional conflict model.

The obtained theoretical results suitable for wide application in multidimensional competitive environments, including military strategy, economic resource allocation, cybersecurity (e.g., distributed defense against concentrated cyberattacks), and biological population dynamics. The mathematical justification of the uniform defense highlights the fundamental strategic value of resource decentralization, risk mitigation, and symmetric distribution in complex, high-dimensional conflicts.

Acknowledgements. This work was partially supported by the Simons Foundation grant SFI-PD-Ukraine-00014586 (O.R.S.) and by the project “Mathematical modeling of complex systems and security-related processes” (No. 0125U000299).

[1] Koshmanenko V., Satur O., On the problem of minimization of losses in a model of dynamical system of conflict on a territory with three regions, *J. Math. Sci.* **212** (2026), no. 2, 111–131.

[2] Koshmanenko V., Spectral theory of conflict, Naukova Dumka, Kyiv, 2016, 288 pp.