Rings of duo-stable range 1

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A ring R is understood as an associative ring with nonzero unit element and $U(R)$ is understood as the group of invertible elements of a ring R and by $M_n(R)$ the set of $n \times n$ matrices over R.

An *elementary matrix* with elements of the ring R is understood as a square matrix of one of the following types: 1) a diagonal matrix with invertible elements on the main diagonal; 2) a matrix that differs from the unit matrix by the presence of any nonzero element outside the main diagonal.

A right (left) Bezout ring is a ring in which every finitely generated right (left) ideal is principal. A *Bezout ring* is a ring which is both right and left Bezout ring.

A ring is said to be a right (left) duo ring if any right (left) ideal of this ring is a 2-sided ideal. If the ring is both left and right duo ring, then it is called *duo ring* [2].

It follows immediately from the definition above that an element a of a ring R is called a *duo-element* if for any $r \in R$ there exists an element $s \in R$ such that $ar = sa$. The set of all duo-elements of the ring R will be denoted by $Duo(R)$.

A ring R is called a *ring of duo-stable range 1* if $aR + bR = R$ implies that there exists $t \in Duo(R)$ such that $a + bt = u$, where $u \in U(R)$.

A ring R is said to have *central stable range 1* [1], if for any $a, b \in R$ satisfying $aR+bR = R$, there exists such $t \in Z(R)$ that $a + bt$ is an invertible element in R.

A ring R is called *a ring with elementary reduction of matrices* [3] in case of an arbitrary matrix over R possesses elementary reduction, i.e. for an arbitrary matrix A over the ring R there exist such elementary matrices over R , $P_1, \dots, P_k$, $Q_1, \dots, Q_s$ of respectful size that

$$P_1 \cdot P_2 \cdots P_k \cdot A \cdot Q_1 \cdot Q_2 \cdots Q_s = diag(\varepsilon_1, \dots, \varepsilon_r, 0, \dots, 0),$$

where $R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_i R$ for any $i = 1, 2, \dots, r-1$.

All other necessary definitions can be found in [1], [3], and [4].

Proposition 1. *Every homomorphic image of a ring of duo-stable range 1 is ring of duo-stable range 1.*

Proposition 2. *Every direct product of rings of duo-stable range 1 is ring of duo-stable range 1.*

Proposition 3. *A ring of central stable range 1 is ring of duo-stable range 1.*

Theorem 1. *A Bezout ring with duo-stable range 1 is a ring with elementary reduction of matrices.*

A ring R is said to be an *EID-ring* [4] if any idempotent matrix E over it has the property of elementary idempotent reduction, i.e., there exist elementary matrices $P_1, \dots, P_k, Q_1, \dots, Q_s$ over R of respectful size such that

$$P_1 \cdot P_2 \cdots P_k \cdot E \cdot (Q_1 \cdot Q_2 \cdots Q_s) = \mathrm{diag}(d_1, d_2, \dots, 0, \dots, 0).$$

Theorem 2. *A Bezout ring with duo-stable range 1 is EID-ring.*

Theorem 3. *Let R is Bezout ring with duo-stable range 1. Then any non-invertible matrix is a product of idempotent matrices.*

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[2] Feller E. H., Properties of primary non-commutative rings, *Trans. Amer. Math. Soc*, **89** (1958), 79–91.

[3] Zabavsky B. V., Rings with elementary reduction matrix, *Ring Theory Conf., Miskolc July*, (1996), 15–20.

[4] Romaniv O. M., Sagan A. V., Firman O. I. Elementary reduction of idempotent matrices, *Applied Problems of Mechanics and Mathematics*, **14** (2016), 7–11 (in Ukrainian).