Poisson limit theorems for peelings in generalized convex hulls

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Convex hull peeling is a construction in multivariate data analysis and stochastic geometry: one forms the convex hull of a sample, removes the sample points which lie on its boundary, and iterates. The successive layers encode the geometry of the sample and are closely related to depth and outlier detection. Recent work of Calka and Quilan established the limit theory for the first layers of random convex hull peelings in a unit ball [1] and simple polytopes [2]. The aim of the present talk is to present the extension of the approach described by Calka and Quilan to families of convex bodies and to apply it for generalized convex hulls of random samples, namely K-hulls, studied in [3].

Let $\mathcal{L} := \{L_i : i \in I\}$ be a non-empty collection of compact convex sets in $\mathbb{R}^d$ with the convex hull

$$\operatorname{conv}(\mathcal{L}) := \operatorname{conv}\left(\bigcup_{i\in I} L_i\right).$$

For $1 \le m < |I|$, the *m-point peeling* of the family $\mathcal{L}$ (or of the convex hull $\operatorname{conv}(\mathcal{L})$) is the set

$$\operatorname{conv}^{[m]}(\mathcal{L}) := \bigcap_{J\subset I, |J|=m} \operatorname{conv}\Big(\bigcup_{i\in I\setminus J} L_i\Big).$$

Thus, if the cardinality $|I|$ is finite, then $\operatorname{conv}^{[m]}(\mathcal{L})$ consists of all points of $\mathbb{R}^d$ which can be represented as convex combinations of points of every subfamily of $\mathcal{L}$ of cardinality $|I| - m$. Alternatively, $\operatorname{conv}^{[m]}(\mathcal{L})$ consists of points that remain in the convex hull $\operatorname{conv}(\mathcal{L})$ no matter which m bodies are removed.

The second type of peeling generalizes that of Calka and Quilan and is defined as follows. A set $L_i \in \mathcal{L}$ is *contributing* to a subfamily $\mathcal{F} \subseteq \mathcal{L}$ if

$$\operatorname{conv}\left(\bigcup_{L_j\in\mathcal{F}} L_j\right) \supsetneq \operatorname{conv}\left(\bigcup_{L_j\in\mathcal{F}\setminus\{L_i\}} L_j\right).$$

Set $\mathcal{L}^{(1)} := \mathcal{L}$. For $m \ge 1$, let $\mathcal{C}_m(\mathcal{L}) \subset I$ be the set of indices of contributing sets of $\mathcal{L}^{(m)}$, and define $I_m := I \setminus \bigcup_{j=1}^{m-1} \mathcal{C}_j(\mathcal{L})$ and $\mathcal{L}^{(m)} := \{L_i : i \in I_m\}$. The *$m$-th recursive convex hull peeling of the family $\mathcal{L}$ (or of the convex hull* $\operatorname{conv}(\mathcal{L})$*)* is

$$\operatorname{conv}_{[m]}(\mathcal{L}) := \operatorname{conv}(\mathcal{L}^{(m)}) = \operatorname{conv}\left(\bigcup_{i\in I_m} L_i\right).$$

Note that when the family $\mathcal{L}$ consists of singletons, the recursive convex hull peeling of $\mathcal{L}$ coincides with the classical convex hull peeling of a point configuration considered by Calka and Quilan [2]. In what follows we use superscripts $[m]$ for the m-point peeling and subscripts $[m]$ for recursive peeling.

We shall now apply the above notions to the K-hulls of random samples as studied in [3]. Let K be a fixed convex body containing the origin in the interior. Recall from [3] that, for a set $A \subset \mathbb{R}^d$, its K-hull is

$$\operatorname{conv}_K(A) := \bigcap_{x\in\mathbb{R}^d \colon A\subset K+x} (K+x).$$

Note that if K is the Euclidean unit ball, then $\operatorname{conv}_K(A)$ is the ball hull of A. Let $\Xi_n = \{\xi_1, \dots, \xi_n\}$ be a random sample of independent points uniformly distributed in K, and define

$$X_n := \bigcap_{i=1}^{n} (K - \xi_i) = K \ominus \Xi_n = K \ominus \operatorname{conv}_K(\Xi_n),$$

where $A \ominus B := \{x \in \mathbb{R}^d : x + B \subseteq A\}$. The polar body to X_n can be written as

$$X_n^o = \left(\bigcap_{i=1}^{n}(K-\xi_i)\right)^o = \operatorname{conv}\left(\bigcup_{i=1}^{n}(K-\xi_i)^o\right) = \operatorname{conv}(\mathcal{L}_n^o),$$

where $\mathcal{L}_n^o := \{(K-\xi_1)^o, \dots, (K-\xi_n)^o\}$.

We define the two peelings of X_n^o as follows

$$(X_n^o)^{[m]} := \operatorname{conv}^{[m]}(\mathcal{L}_n^o) \quad \text{and} \quad (X_n^o)_{[m]} := \operatorname{conv}_{[m]}(\mathcal{L}_n^o).$$

Our main results establish the convergence of properly rescaled peelings $(X_n^o)^{[m]}$ and $(X_n^o)_{[m]}$. The limiting objects are defined through a Poisson hyperplane tessellation. Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$. Let $\mathcal{P}_K = \{(t_i, u_i),\ i \geq 1\}$ be a Poisson process on $(0,\infty) \times S^{d-1}$ with intensity measure equal to the product of $V_d(K)^{-1}$, Lebesgue measure on $(0,\infty)$, and the surface area measure $S_{d-1}(K, \cdot)$. The associated half-spaces $H_{u_i}^-(t_i)$ generate a Poisson tessellation of $\mathbb{R}^d$, and its zero cell is

$$Z := \bigcap_{(t_i,u_i)\in\mathcal{P}_K} H_{u_i}^-(t_i),$$

where $H_u^-(t) := \{x \in \mathbb{R}^d : \langle x, u\rangle \leq t\}$. Let Π_K be the image of $\mathcal{P}_K$ under $(t,u) \mapsto t^{-1}u$. Then

$$Z^o = \operatorname{conv}(\Pi_K).$$

This representation allows us to define two peelings of Z^o via $Z^o_{[m]} = \operatorname{conv}_{[m]}(\Pi_K)$ and $(Z^o)^{[m]} = \operatorname{conv}^{[m]}(\Pi_K)$. Let $\mathcal{K}_d^{(0)}$ be the space of convex bodies on $\mathbb{R}^d$ endowed with the Hausdorff distance.

Theorem 1 (m-point peeling). *Assume that the convex body K is strictly convex. Then, for every $m \in \mathbb{N}$, the random vector of convex bodies $\big(n^{-1}(X_n^o)^{[1]}, \dots, n^{-1}(X_n^o)^{[m]}\big)$ converges in distribution, as $n \to \infty$, on the space $(\mathcal{K}_d^{(0)})^m$ endowed with the product topology to $\big((Z^o)^{[1]}, \dots, (Z^o)^{[m]}\big)$.*

Theorem 2 (Convex hull peeling). *For every $m \in \mathbb{N}$, the random vector of convex bodies $\big(n^{-1}(X_n^o)_{[1]}, \dots, n^{-1}(X_n^o)_{[m]}\big)$ converges in distribution, as $n \to \infty$, on the space $(\mathcal{K}_d^{(0)})^m$ endowed with the product topology to $\big(Z^o_{[1]}, \dots, Z^o_{[m]}\big)$.*

[1] Calka P., Quilan G., Limit theory for the first layers of the random convex hull peeling in the unit ball, *Probability Theory and Related Fields* **187** (2023), 3–4, 1037–1091.

[2] Calka P., Quilan G., Limit theory for the first layers of the random convex hull peeling in a simple polytope, *Advances in Mathematics* **483** (2025), 110670.

[3] Marynych A., Molchanov I., Facial structure of strongly convex sets generated by random samples, *Advances in Mathematics* **395** (2022), 108086.