TEMPERED FRACTIONAL ORNSTEIN–UHLENBECK PROCESSES

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Tempered fractional Brownian motion is a modification of fractional Brownian motion obtained by introducing exponential tempering into the Mandelbrot–Van Ness representation. It preserves the local fractional behavior and weakens dependence at large lags.

Let $H > 0$, $\lambda > 0$, $\theta > 0$, $x_+ = \max\{x, 0\}$, and let $W = \{W_x, x \in \mathbb{R}\}$ be a two-sided Wiener process. The parameter $k \in \{\mathrm{I}, \mathrm{II}\}$ indicates the type of tempered fractional Brownian motion: the first kind [4] or the second kind [5]. In both cases

$$B^k_{H,\lambda}(t) = \int_{\mathbb{R}} g^k_{H,\lambda,t}(x)\, dW_x,$$

where

$$g^{\mathrm{I}}_{H,\lambda,t}(x) = e^{-\lambda(t-x)_+}(t-x)_+^{H-1/2} - e^{-\lambda(-x)_+}(-x)_+^{H-1/2},$$
$$g^{\mathrm{II}}_{H,\lambda,t}(x) = g^{\mathrm{I}}_{H,\lambda,t}(x) + \lambda \int_0^t e^{-\lambda(u-x)_+}(u-x)_+^{H-1/2}\, du.$$

We study stationary and non-stationary Ornstein–Uhlenbeck processes driven by $B^k_{H,\lambda}$; classical fractional Ornstein–Uhlenbeck processes were considered in [1]. These processes solve the Langevin-type equation $dX_t = -\theta X_t\, dt + dB^k_{H,\lambda}(t)$. The stationary solution has the form

$$Y^k_t = B^k_{H,\lambda}(t) - \theta e^{-\theta t} \int_{-\infty}^t e^{\theta s} B^k_{H,\lambda}(s)\, ds, \qquad t \ge 0,$$

whereas the non-stationary solution with initial condition $X^k_0 = 0$ is given by

$$X^k_t = B^k_{H,\lambda}(t) - \theta e^{-\theta t} \int_0^t e^{\theta s} B^k_{H,\lambda}(s)\, ds.$$

The process Y^k_t admits the Wiener integral representation $Y^k_t = \int_{\mathbb{R}} h^k(t-x)\, dW_x$ with kernel

$$h^k(z) = \left(e^{-\lambda z} z^{H-1/2} - a_k e^{-\theta z} \int_0^z e^{(\theta-\lambda)y} y^{H-1/2}\, dy \right) \mathbf{1}_{(0,\infty)}(z),$$

where $\mathbf{1}_{(0,\infty)}$ is the indicator function, $a_{\mathrm{I}} = \theta$, and $a_{\mathrm{II}} = \theta - \lambda$. Hence

$$\operatorname{cov}(Y^k_t, Y^k_s) = r_k(t-s), \qquad r_k(\tau) = \int_{\mathbb{R}} h^k(\tau+u) h^k(u)\, du.$$

Since Y^k is centered Gaussian and the covariance depends only on the time difference, the process Y^k is stationary.

For the autocovariance functions $r_k(\tau)$, $k \in \{\mathrm{I}, \mathrm{II}\}$, the following asymptotic relations are obtained as $\tau \to \infty$:

	$r_{\mathrm{I}}(\tau)$	$r_{\mathrm{II}}(\tau)$
$\theta < \lambda$	$\sim -\dfrac{\theta \Gamma(H+\frac12)^2}{2(\lambda^2-\theta^2)^{H+\frac12}} e^{-\theta\tau}$	$\sim \dfrac{\Gamma(H+\frac12)^2}{2\theta(\lambda^2-\theta^2)^{H-\frac12}} e^{-\theta\tau}$
$\theta = \lambda$	$\sim -\dfrac{\Gamma(H+\frac12)}{(H+\frac12)2^{H+\frac32}\theta^{H-\frac12}} e^{-\theta\tau}\tau^{H+\frac12}$	$\sim \dfrac{\Gamma(H+\frac12)}{2^{H+\frac12}\theta^{H+\frac12}} e^{-\theta\tau}\tau^{H-\frac12}$
$\theta > \lambda$	$\sim -\dfrac{\Gamma(H+\frac12)}{2^{H+\frac12}\lambda^{H-\frac32}(\theta^2-\lambda^2)} e^{-\lambda\tau}\tau^{H-\frac12}$	$\sim \dfrac{(H-\frac12)\Gamma(H+\frac12)}{2^{H-\frac12}\lambda^{H-\frac12}(\theta-\lambda)(\theta+\lambda)} e^{-\lambda\tau}\tau^{H-\frac32},\ H \ne \frac12$

In the exceptional case $k = \mathrm{II}$, $H = 1/2$, $\theta > \lambda$, one has the exact identity $r_{\mathrm{II}}(\tau) = (2\theta)^{-1}e^{-\theta\tau}$, $\tau \geq 0$. Consequently, $r_k(\tau) \to 0$ and Y^k is ergodic.

For drift parameter estimation with known H and λ, we use an ergodic-type approach close to [2,3]. Let $\psi_k(\theta) = \mathbb{E}_\theta(Y_0^k)^2$. The second moments have the spectral representations

$$\psi_{\mathrm{I}}(\theta) = \frac{\Gamma(H+1/2)^2}{\pi} \int_0^\infty \frac{\omega^2 \, d\omega}{(\theta^2+\omega^2)(\lambda^2+\omega^2)^{H+1/2}},$$

$$\psi_{\mathrm{II}}(\theta) = \frac{\Gamma(H+1/2)^2}{\pi} \int_0^\infty \frac{d\omega}{(\theta^2+\omega^2)(\lambda^2+\omega^2)^{H-1/2}}.$$

The functions ψ_k are continuous and strictly decreasing on $(0,\infty)$; hence θ is identifiable from the variance.

For X^k, the ergodic limits

$$\frac{1}{T}\int_0^T (X_t^k)^2 \, dt \to \psi_k(\theta), \qquad \frac{1}{N}\sum_{j=1}^N (X_{jh}^k)^2 \to \psi_k(\theta)$$

hold almost surely as $T \to \infty$ and $N \to \infty$. This yields the estimators

$$\widehat{\theta}_T^k = \psi_k^{-1}\left(\frac{1}{T}\int_0^T (X_t^k)^2 \, dt\right), \qquad \widehat{\theta}_{N,h}^k = \psi_k^{-1}\left(\frac{1}{N}\sum_{j=1}^N (X_{jh}^k)^2\right).$$

They are strongly consistent and asymptotically normal:

$$\sqrt{T}\left(\widehat{\theta}_T^k - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_k^2}{(\psi_k'(\theta))^2}\right), \qquad \sqrt{N}\left(\widehat{\theta}_{N,h}^k - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_{k,h}^2}{(\psi_k'(\theta))^2}\right),$$

where $\xrightarrow{d}$ denotes convergence in distribution,

$$\sigma_k^2 = 2\int_{\mathbb{R}} r_k(\tau)^2 \, d\tau, \qquad \sigma_{k,h}^2 = 2\sum_{m\in\mathbb{Z}} r_k(mh)^2.$$

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