Extension of birth-and-death semi-Markov processes beyond the moment of explosion

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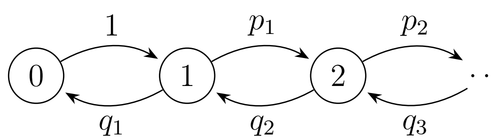
Let $(X_k)_{k\geq 0}$ be a birth-and-death Markov chain on the state space $\mathbb{N}_0$ with transition probabilities $p_{i,i+1} = p_i$, $p_{i,i-1} = q_i$, where $p_i, q_i > 0$, $p_i + q_i = 1$ for $i \geq 1$, and $p_{0,1} = p_0 = 1$. Its diagram is given below.



Consider a sequence of non-negative random variables $\{\tau_i\}_{i=0}^{\infty}$. For each τ_i, we introduce a sequence of its independent copies $\{\tau_i^k\}_{k=0}^{\infty}$. Suppose that all these sequences and the Markov chain (X_k) are jointly independent. Define the jump moments recurrently: $T_0 = 0$, $T_{k+1} = T_k + \tau_{X_k}^k$, $k \geq 0$.

Definition 1. The process $X(t) = X_k$ for $t \in [T_k, T_{k+1})$, $k \geq 0$, is called a *semi-Markov process* (or continuous-time random walk).

Denote $\sigma_n = \inf\{t \geq 0 \mid X(t) = n\}$. Then $\sigma = \lim_{n\to\infty} \sigma_n$ is called the *moment of explosion*. If $\sigma < \infty$ almost surely, the process is said to *explode*. Exact analytical conditions for explosion were obtained by the authors in [1].

If the process explodes, its trajectories are well-defined only on the finite interval $[0, \sigma)$. Our goal is to construct an extension $\widetilde{X}(t)$ for all $t \geq 0$, which behaves as $X(t)$ on $\mathbb{N}_0$ and has an "instantaneous reflection" at infinity. To construct such an extension, we consider a sequence of processes $\{X^{(n)}\}_{n\geq 1}$ having the same characteristics as X, but strictly reflected at the boundary state n. To formalize the reflection at infinity, we equip the one-point compactification $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$ with the metric $d(i,j) = \left|\frac{1}{i+1} - \frac{1}{j+1}\right|$, where $\frac{1}{\infty+1} = 0$.

Theorem 1. *Suppose that the process X explodes and the following condition holds:*

$$\sum_{i=1}^{\infty} (1 - \mathsf{E}e^{-\tau_i}) \frac{p_0 \cdots p_{i-1}}{q_1 \cdots q_i} < \infty.$$

Then the sequence of reflected processes $\{X^{(n)}\}_{n\geq 1}$ converges in probability in the J_1-topology of the Skorokhod space $D([0,\infty), \overline{\mathbb{N}}_0)$ to a unique limit process. This process is the desired extension of X.

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[1] Pilipenko A., Tkachenko V., Explosion and implosion of birth-and-death continuous-time random walks, *arXiv preprint* (2025), 17 pp., arXiv:2511.11076.