On the Kiguradze theorem for linear boundary-value problems

O. B. Pelekhata[1], **N. V. Reva**[2]
[1,2]National technical university of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute",
Kyiv, Ukraine
rnadiia-fmf@lll.kpi.ua, pelehataob2015@gmail.com

The question of finding the conditions for the convergence of solutions of systems of ordinary differential equations arises in many problems of modern analysis and its applications. This question has been deeply investigated in the case of the solutions of Cauchy's problems for the system of first-order differential equations. A more complicated case of linear boundary-value problems was studied by I.T. Kiguradze [1] and his followers [2, 3].

On a finite interval $(a, b) \subset \mathbb{R}$, we consider the systems of $m \in \mathbb{N}$ linear differential equations of the first order

$$y'(t,n) + A(t,n)y(t,n) = f(t,n) \tag{1}$$

with inhomogeneous boundary conditions

$$B(n)y(\cdot,n) = c(n), \quad n \in \mathbb{N} \cup \{0\}. \tag{2}$$

We suppose that the linear continuous operator $B(n) : C\left([a,b];\mathbb{C}^m\right) \to \mathbb{C}^m$, matrix-valued functions $A(\cdot,n) \in L_1([a,b];\mathbb{C}^{m\times m})$, the vector-valued functions $f(\cdot,n) \in L_1([a,b];\mathbb{C}^m)$, and the vectors $c(n) \in \mathbb{C}^m$ are given.

The solution of the system of differential equations (1) is understood as a vector-valued function $y(\cdot) \in W_1^1([a,b];\mathbb{C}^m)$ absolutely continuous on the compact interval $[a,b]$ satisfying the vector equation (1) almost everywhere. The inhomogeneous boundary condition (2) is correctly defined on the solutions of system of (1) and covers all classical types of boundary conditions. For unique solvability for arbitrary right-hand sides, it is necessary and sufficient to guarantee that the corresponding homogeneous boundary-value problem has only trivial solution.

Assume that the solution of problem (1), (2), with $n = 0$, is uniquely defined. Then the following problems are of high importance:

Under what conditions imposed on the left-hand sides of problems (1), (2) their solutions $y(\cdot,n)$ exist and are unique for sufficiently large $n \in \mathbb{N}$? What additional conditions imposed on the left- and right-hand sides of problems (1), (2) guarantee the limit equality

$$\|y(\cdot,n) - y(\cdot,0)\|_\infty \to 0, \quad n \to \infty, \tag{3}$$

where $\|\cdot\|_\infty$ – sup-norm on the compact interval $[a,b]$.

For the first time, these problems were investigated by Kiguradze [1] in the case of real-valued functions.

We introduce the notation:

$$R_A(\cdot,n) := A(\cdot,n) - A(\cdot,0) \in L_1([a,b];\mathbb{C}^{m\times m}),$$

$$F(\cdot,n) := \begin{pmatrix} f_1(\cdot,n) & 0 & \dots & 0 \\ f_2(\cdot,n) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_m(\cdot,n) & 0 & \dots & 0 \end{pmatrix} \in L_1([a,b];\mathbb{C}^{m\times m}), \quad R_F(\cdot,n) = F(\cdot,n) - F(\cdot,0),$$

$$R_F^{\vee}(t,n) := \int_a^t R_F(s,n)ds, \quad R_A^{\vee}(t,n) := \int_a^t R_A(s,n)ds,$$

Let $\|\cdot\|_1$ denote the norm in Lebesgue space of vector-valued functions (matrix-valued functions) on the interval $[a,b]$.

Further we assume that all asymptotic relations are considered as $n \to \infty$.

Theorem 1 (Kiguradze [1]). *Suppose that*

(0) *the homogeneous boundary-value problem (1), (2), with $n = 0$, has only the trivial solution;*

(I) $\|R_A^{\vee}(\cdot,n)\|_\infty \to 0$*;*

(II) $\|R_A(\cdot,n)\|_1 = O(1)$*;*

(III) $B(n)y \to B(0)y, \quad y(\cdot) \in C([a,b];\mathbb{C}^m)$.

Then, for sufficiently large n, problem (1), (2) possesses a unique solution. In addition, if the right-hand sides of problems satisfy the following conditions

(IV) $c(n) \to c(0)$*;*

(V) $\|R_F^{\vee}(\cdot,n)\|_\infty \to 0$,

then the unique solutions of problems (1), (2) satisfy the limit equality (3).

The examples show that all conditions of the Kiguradze Theorem are essential and none of them can be omitted. However, some conditions can be weakened.

Denote by $\mathcal{M}^m := \mathcal{M}(a,b;m), m \in \mathbb{N}$ class of sequences of the matrix functions $R(\cdot,n) : \mathbb{N} \to L_1([a,b];\mathbb{C}^{m\times m})$, such that the solution $Z(\cdot,n)$ of Cauchy problem

$$Z'(\cdot,n) + R(\cdot,n)Z(\cdot,n) = O, \quad Z(a,n) = I_m$$

satisfies the limit equality

$$\|Z(\cdot,n) - I_m\|_\infty \to 0,$$

where I_m is identity $(m\times m)$–matrix.

Put

$$A_F(\cdot,n) := \begin{pmatrix} A(\cdot,n) & F(\cdot,n) \\ O_m & O_m \end{pmatrix} \in L_1([a,b];\mathbb{C}^{2m\times 2m}), \quad R_{A_F}(\cdot,n) := A_F(\cdot,n) - A_F(\cdot,0)$$

where O_m is zero $(m\times m)$–matrix.

Theorem 2. *In Kiguradze Theorem, conditions (I), (II) can be replaced by one condition*

$$R_A(\cdot,n) \in \mathcal{M}^m,$$

if condition (V) is replaced by the following condition

$$R_{A_F}(\cdot,n) \in \mathcal{M}^{2m}.$$

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