HIGHER-ORDER MOMENTS OF THE WINDING NUMBER OF THE GAUSSIAN RANDOM FIELD ALONG A SMOOTH CURVE

H. V. Navrostkyi

[1]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

navrotskiy1992@gmail.com

Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field. $L : [0,1] \to \mathbb{R}^2$ be a continuous curve, such that Ψ does not turn to 0 on the curve L . Then

Definition 1. (p. 4 in [1]) The winding number $\gamma(\Psi, L)$ of the field Ψ along the curve L is the number $\frac{1}{2\pi}(\phi(1) - \phi(0))$, where ϕ is the continuous branch of the angular function of Ψ .

Theorem 1. *(p. 23 in [1]) The winding number of a continuous vector field ξ on a boundary of a region $B \subset \mathbb{R}^2$ can be calculated as follows*

$$\gamma(\xi, B) = \sum_{u \in B : \xi(u) = 0} ind(\xi)(u)$$

if ξ has no null vectors on the boundary of B.

Here $\operatorname{ind}(\xi)(u)$ denotes the index of the field ξ in the point u, i.e. the rotation of the field on a small enough circle with center at the point u.

If the matrix of derivatives of ξ is non-degenerate in the point u, the index can be calculated as (p. 61 in [1])

$$\operatorname{ind}(\xi)(u) = \operatorname{sign}(\det(\xi'(u))).$$

Let us define the factorial power of $k-$th order of a random variable X as follows

$$X_{(k)} = X(X-1)\dots(X-k+1) .$$

We obtained the following identity for the higher-order factorial moments of the winding number of random field.

Theorem 2. *Let the following conditions hold :*

a) ξ and $\frac{\partial \xi_i}{\partial t_j}$ are continuous on B with probability 1 and have bounded moments of the order $2k$ on B.

b) $\forall t \in B$, the probability density $p_{\xi(t)}(x)$ of the random vector $\xi(t)$ is continuous at the point $x = 0$.

b′) For all $\tilde{t} \in B_0 = \{(t_1, \dots t_k) \in B^k : t_i \neq t_j, i \neq j\}$, the probability density $p_{\tilde{\xi}(\tilde{t})}(x_1, \dots x_k)$ of the random vector $\tilde{\xi}(\tilde{t}) = (\xi(t_1), \xi(t_2), \dots, \xi(t_k))$ is continuous at $(x_1, \dots x_k) = 0$.

c) For all $t \in B$, the conditional probability density $p_t(x|\nabla\xi(t))$ of the random vector $\xi(t)$ exists, is bounded, and is continuous at $x = 0$.

c′) For all $\tilde{t} \in B_0$, the conditional probability density $p_{\tilde{t}}(\tilde{x}|\nabla\tilde{\xi}(\tilde{t}))$ of the random vector $\tilde{\xi}(\tilde{t})$ exists, is continuous, and is bounded at $\tilde{x} = (0, \dots 0)$.

d) For all $t \in B$, the conditional probability density $p_t(z|\xi(t) = x)$ of the random value $\nabla\xi(t)$ is continuous in z and x.

d′) For all $\tilde{t} \in B_0$ the conditional probability density $p_{\tilde{t}}(z|\tilde{\xi}(\tilde{t}) = \tilde{x})$ of the value $\nabla\tilde{\xi}(\tilde{t})$ is continuous in z and x.

e) The modulus of continuity on B of each of the components of ξ, $\nabla\xi$ satisfies the relation $P(\omega(\eta) > \epsilon) = o(\eta^{2k})$, when $\eta \downarrow 0$ for all $\epsilon > 0$.

Also let us denote as

$$N_+ = \#\{r \in B : \xi(r) = 0, det(\xi'(r)) > 0\}$$

and respectively

$$N_- = \#\{r \in B : \xi(r) = 0, det(\xi'(r)) < 0\}$$

Then

$$\sum_{s_1 \dots s_k \in \{1,-1\}} s_1 \dots s_k \mathbb{E}((N_+)_{(\sum_{i=1}^k s_i + k)/2} (N_-)_{(k - \sum_{i=1}^k s_i)/2}) =$$

$$= \int_{B^k} \mathbb{E}(\prod_{j=1}^k det(D(t_j)) | \xi(t_1) = \dots = \xi(t_k) = 0) p_{\tilde{\xi}(\tilde{t})}(0) dt_1 \dots dt_k$$

where $p_{\tilde{\xi}(\tilde{t})}(0)$ *is the probability density of the random vector* $\tilde{\xi}(\tilde{t})$ *at the point* 0*,* $D(t_j)$ *is the matrix of partial derivatives of the field* ξ *at the point* t_j*.*

Example 1. As an example of the random field that satisfies conditions a)-e) of the theorem for all $k \in \mathbb{N}$, we consider the homogeneous centered Gaussian planar random field ξ with independent components and the following covariance function of components:

$$\mathbb{E}\xi_i(u)\xi_i(v) = e^{-||u-v||^2},\ i = 1, 2$$

The condition e) is satisfied (p. 268 in book [2]).

[1] Krasnoselskii M., Perov A., Povolockii A., Zabreiko P., *Plane vector fields*, Academic Press, New York, 1966, 242 pp.

[2] Adler R. J., Taylor J. E., *Random Fields and Geometry*, Springer, New York, 2007, 472 pp.