Homotopy structure of orbits of smooth functions lifted to an orientable double cover

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Let M be a surface (smooth compact 2-manifold) and $f\colon M \to \mathbb{R}$ be a smooth function. By smooth we mean $\mathcal{C}^\infty$. We study the homotopy types of orbits $\mathcal{O}(f)$ with respect to the natural right action of the smooth diffeomorphisms group on the space of smooth functions $\mathcal{C}^\infty(M,\mathbb{R}) \times \mathcal{D}(M) \to \mathcal{C}^\infty(M,\mathbb{R})$, given by $(f,h) \mapsto f \circ h$.

Consider the subclass $\mathcal{F}(M,\mathbb{R}) \subset \mathcal{C}^\infty(M,\mathbb{R})$ of functions that, in particular, contains Morse functions, which are dense in $\mathcal{C}^\infty(M,\mathbb{R})$. It is well-known (see [3]) that, for each $f \in \mathcal{F}(M,\mathbb{R})$, the homotopy types of $\mathcal{O}(f)$ are weak, i.e., they are completely described by the collection of the homotopy groups $\pi_k\mathcal{O}(f)$, $k \geq 1$. Only a few cases yield non-trivial higher homotopy groups $\pi_k\mathcal{O}(f)$, $k \geq 2$, while the structure of the fundamental group $\pi_1\mathcal{O}(f)$ is highly non-trivial and is of the main interest. Cases of certain surfaces M are well-studied and well-understood, while some are still remaining an open question, see [2] and [3].

Let $\tilde{N}$ be an orientable surface with a smooth orientation-reversing involution $\tau\colon \tilde{N} \to \tilde{N}$ without fixed points, and $N = \tilde{N}/\tau$ be the quotient, a non-orientable surface. Then there is a smooth double covering map $p\colon \tilde{N} \to N$, and the lifted function $\tilde{f} = f \circ p$, which lies in the class $\mathcal{F}(\tilde{N},\mathbb{R})$. One can uniquely map each $h \in \mathcal{D}(N)$ to the orientation-preserving lift $\tilde{h} \in \mathcal{D}(\tilde{N})$ such that $p \circ \tilde{h} = h \circ p$. This lifting induces a well-defined mapping of orbits $\theta\colon \mathcal{O}(f) \to \mathcal{O}(\tilde{f})$.

Theorem 1. *Map θ is π_1-injective, i.e., induced map $\theta_1\colon \pi_1\mathcal{O}(f) \to \pi_1\mathcal{O}(\tilde{f})$ is a group monomorphism.*

Theorem 2. *Assume Euler characteristic $\chi(N) < 0$, and function $f \in \mathcal{F}(N,\mathbb{R})$.*

1. *There exists (see e.g. [1]) a finite collection $\{S_\alpha\}$ of subsurfaces of N whose elements are diffeomorphic either to disks D_i, cylinders C_j, or Mobius bands M_k, such that*

$$\begin{aligned}\pi_1\mathcal{O}(f) &\cong \prod_\alpha \pi_1\mathcal{O}(f|_{S_\alpha}) \\ &\cong \prod_i \pi_1\mathcal{O}(f|_{D_i}) \times \prod_j \pi_1\mathcal{O}(f|_{C_j}) \times \prod_k \pi_1\mathcal{O}(f|_{M_k}).\end{aligned}$$

2. *There exists a finite collection $\{\tilde{S}_\beta\}$ of subsurfaces of $\tilde{N}$ whose elements are diffeomorphic either to disks $\tilde{D}_u$, or cylinders $\tilde{C}_v$. Moreover, there is a subcollection of cylinders $\{\tilde{C}_k\} \subset \{\tilde{C}_u\}$ which are in one-to-one correspondence with the collection of Mobius bands $\{M_k\}$, such that*

$$\begin{aligned}\pi_1\mathcal{O}(\tilde{f}) &\cong \prod_\beta \pi_1\mathcal{O}(f|_{\tilde{S}_\beta}) \\ &\cong \prod_u \pi_1\mathcal{O}(\tilde{f}|_{\tilde{D}_u}) \times \prod_v \pi_1\mathcal{O}(\tilde{f}|_{\tilde{C}_v}) \\ &\cong \prod_i (\pi_1\mathcal{O}(f|_{D_i}))^2 \times \prod_j (\pi_1\mathcal{O}(f|_{C_j}))^2 \times \prod_k \pi_1\mathcal{O}(\tilde{f}|_{\tilde{C}_k}).\end{aligned}$$

In particular, $\#\{\tilde{D}_u\} = 2\#\{D_i\}$ and $\#\{\tilde{C}_v\} = 2\#\{C_j\} + \#\{M_k\}$.

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[1] Mazhar B., Maksymenko S., Smooth functions that split a Klein bottle into two Möbius bands, *preprint* (2025), arXiv:2508.19636.

[2] Kuznietsova I., Maksymenko S., Deformational symmetries of smooth functions on non-orientable surfaces. *Topol. Methods Nonlinear Anal.* (2025), 1–43.

[3] Maksymenko S., Deformations of functions on surfaces by isotopic to the identity diffeomorphisms, *Topology Appl.*, **282** (2020), 107312.