

POINT SYMMETRY PSEUDOGROUPS OF LINEAR SCHRÖDINGER EQUATIONS WITH COMPLEX TIME-INDEPENDENT POTENTIALS

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The computation of the maximal Lie invariance algebra of a system of differential equations is a quite algorithmic problem, and there are a number of specialized packages for this purpose in various computer algebra systems. Nevertheless, at least a part of these packages sometimes miss a part of Lie symmetries, produce incorrect Lie symmetries, and the situation becomes worse in the course of studying a class of systems of differential equations. Finding the complete point symmetry groups of a system of differential equations, including its discrete symmetries, is not as algorithmic as finding its maximal Lie invariance algebra. Moreover, the problem of classifying point symmetries for a class of such systems has not properly been addressed in the literature. Using results of [1], we solve this problem for the class \mathcal{F}' of (1+1)-dimensional linear Schrödinger equations \mathcal{F}'_V with time-independent complex-valued potentials V , which are of the form

$$i\psi_t + \psi_{xx} + V(x)\psi = 0, \tag{1}$$

where ψ is an unknown complex-valued function of two real independent variables (t, x) and V is an arbitrary smooth complex-valued potential depending on x .

Theorem. A complete list of $G_{\mathcal{F}'}$ -inequivalent cases for the point-symmetry pseudogroups G_V of equations \mathcal{F}'_V from the class \mathcal{F}' , where $G_{\mathcal{F}'}$ denotes the equivalence group of the class \mathcal{F}' , is exhausted by the following cases, where for each case we present the corresponding potential(s) V and the point transformations constituting the group G_V .

0. general $V(x)$: the compositions of the transformations $\tilde{t} = t + \lambda$, $\tilde{x} = x$, $\tilde{\psi} = \sigma(\psi + \Lambda)$ with the (discrete) transformations of the form $\tilde{t} = \lambda't$, $\tilde{x} = |\lambda'|^{1/2}x + \nu$, $\tilde{\psi} = e^{\sigma't}\hat{\psi}$ such that $V(\tilde{x}) = \hat{V}(x)/|\lambda'| - i\sigma'/\lambda'$,
 1. $V(x) = i\alpha x - x^2$: $\tilde{t} = \varepsilon t + \lambda$, $\tilde{x} = \varepsilon x + 2\kappa \cos(2t + \nu)$,
 $\tilde{\psi} = \sigma \exp(\kappa \sin(2t + \nu)(-2ix - 2i\varepsilon\kappa \cos(2t + \nu) - \varepsilon\alpha))(\hat{\psi} + \hat{\Lambda})$,
 2. $V(x) = i\alpha x + x^2$: $\tilde{t} = \varepsilon t + \lambda$, $\tilde{x} = \varepsilon x + 2\kappa e^{2t} + 2\nu e^{-2t}$,
 $\tilde{\psi} = \sigma \exp((\kappa e^{2t} - \nu e^{-2t})(2ix + 2i\varepsilon\kappa e^{2t} + 2i\varepsilon\nu e^{-2t} - \varepsilon\alpha))(\hat{\psi} + \hat{\Lambda})$,
 - 3a. $V(x) = ix$: $\tilde{t} = \varepsilon t + \lambda$, $\tilde{x} = \varepsilon x + 2\kappa t + \nu$, $\tilde{\psi} = \sigma \exp(i\kappa x + \varepsilon i\kappa^2 t - \varepsilon\kappa t^2 - \varepsilon\nu t)(\hat{\psi} + \hat{\Lambda})$,
 - 3b. $V(x) = i\alpha x + x$: $\tilde{t} = \varepsilon t + \lambda$, $\tilde{x} = \varepsilon x + (1 - \varepsilon)t^2 + 2(\kappa + \varepsilon\lambda)t + \lambda^2 + \nu$,
 $\tilde{\psi} = \sigma \exp(i(\varepsilon t + \lambda)(\varepsilon x - \varepsilon t^2 + 2\kappa t + \nu) + i\kappa(x - t^2) - \varepsilon t(ix + \kappa\alpha t + \alpha\nu - i\kappa^2) + \frac{1}{3}(2i - \alpha)(\varepsilon t + \lambda)^3 + \frac{1}{3}(i\varepsilon + \alpha)t^3)(\hat{\psi} + \hat{\Lambda})$,
 - 4a. $V(x) = \mu/x^2$: as in Case 5a
 - 4b. $V(x) = \mu/x^2 - x^2$: as in Case 5b
 - 4c. $V(x) = \mu/x^2 + x^2$: as in Case 5c
- with $\kappa = \nu = 0$ if $\mu \in \mathbb{R} \setminus \{0\}$ and
 in addition with $\varepsilon' = 1$ if $\mu \in \mathbb{C} \setminus \mathbb{R}$,

5a. $V(x) = 0$: $\tilde{t} = \frac{\lambda_1 t + \lambda_2}{\lambda_3 t + \lambda_4}$, $\tilde{x} = \frac{x + \kappa t + \nu}{\lambda_3 t + \lambda_4}$,
 $\tilde{\psi} = \sigma \sqrt{|\lambda_3 t + \lambda_4|} \exp\left(-i\varepsilon' \frac{\lambda_3 x^2 - (\kappa\lambda_4 - \nu\lambda_3)(2x + \kappa t + \nu)}{4(\lambda_3 t + \lambda_4)}\right) (\hat{\psi} + \hat{\Lambda})$,

5b. $V(x) = -x^2$: $\tilde{t} = \frac{1}{2} \arctan \frac{2\rho}{\varsigma}$, $\tilde{x} = \frac{2x + \vartheta}{\varsigma(1 + 4\rho^2/\varsigma^2)^{1/2}}$,
 $\tilde{\psi} = \sigma(\varsigma^2 + 4\rho^2)^{1/4} \exp\left(\frac{-i\rho(2x + \vartheta)^2}{\varsigma(\varsigma^2 + 4\rho^2)} - i\frac{\varepsilon'\varsigma_t}{4\varsigma}x^2 + i\varepsilon'(\kappa\lambda_4 - \nu\lambda_3)\frac{4x + \vartheta}{4\varsigma}\right) (\hat{\psi} + \hat{\Lambda})$
 with $\rho := \lambda_1 \sin 2t + 2\lambda_2 \cos 2t$, $\varsigma := \lambda_3 \sin 2t + 2\lambda_4 \cos 2t$, $\vartheta := \kappa \sin 2t + 2\nu \cos 2t$,

5c. $V(x) = x^2$: $\tilde{t} = \frac{1}{4} \ln \left| 4\frac{\rho}{\varsigma} \right|$, $\tilde{x} = \frac{4x + \vartheta}{2\varsigma|\rho/\varsigma|^{1/2}}$,
 $\tilde{\psi} = \sigma|\rho\varsigma|^{1/4} \exp\left(-i\frac{(4x + \vartheta)^2}{8|\rho\varsigma|} - \frac{i\varepsilon'\varsigma_t}{4\varsigma \operatorname{sgn}(\rho\varsigma)}x^2 + i\varepsilon'(\kappa\lambda_4 - \nu\lambda_3)\frac{8x + \vartheta}{4\varsigma \operatorname{sgn}(\rho\varsigma)}\right) (\hat{\psi} + \hat{\Lambda})$
 with $\rho := \lambda_1 e^{2t} + 4\lambda_2 e^{-2t}$, $\varsigma := \lambda_3 e^{2t} + 4\lambda_4 e^{-2t}$, $\vartheta := \kappa e^{2t} + 4\nu e^{-2t}$,

5d. $V(x) = x$: $\tilde{t} = \frac{\lambda_1 t + \lambda_2}{\lambda_3 t + \lambda_4}$, $\tilde{x} = \frac{x - t^2 + \kappa t + \nu}{\lambda_3 t + \lambda_4} + \left(\frac{\lambda_1 t + \lambda_2}{\lambda_3 t + \lambda_4}\right)^2$,
 $\tilde{\psi} = \sigma \sqrt{|\lambda_3 t + \lambda_4|} \exp\left(i\frac{\lambda_1 t + \lambda_2}{(\lambda_3 t + \lambda_4)^2}(x - t^2 + \kappa t + \nu) + \frac{2}{3}i\left(\frac{\lambda_1 t + \lambda_2}{\lambda_3 t + \lambda_4}\right)^3\right)$
 $\times \exp\left(-i\varepsilon' \frac{\lambda_3(x - t^2)^2 - (\kappa\lambda_4 - \nu\lambda_3)(2x - 2t^2 + \kappa t + \nu)}{4(\lambda_3 t + \lambda_4)} - i\varepsilon' t x + i\varepsilon' \frac{t^3}{3}\right) (\hat{\psi} + \hat{\Lambda})$,

where $\alpha, \lambda', \lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \kappa$ and ν are arbitrary real constants with $\lambda_1\lambda_4 - \lambda_2\lambda_3 := \varepsilon' = \pm 1$, $\lambda' \neq 0$ and $\alpha \neq 0$, σ and σ' are arbitrary complex constants with $\sigma \neq 0$, $\varepsilon = \pm 1$, and $\Lambda = \Lambda(t, x)$ is an arbitrary solution of the equation \mathcal{F}'_V , hat over a complex value denotes the same value or its complex conjugate if the derivative of the t -component of this transformation or, equivalently, the relevant constant λ', ε or ε' is positive or negative, respectively.

The point-symmetry pseudogroup G_V of each equation \mathcal{F}'_V splits over the pseudogroup G_V^{lin} of its transformations of linear superposition of solutions, $G_V = G_V^{\text{ess}} \times G_V^{\text{lin}}$. Here the subgroup G_V^{ess} of G_V consists of the transformations of the corresponding form from the theorem with $\Lambda = 0$ and with the natural domains in the entire space with the coordinates (t, x, ψ, ψ^*) . This subgroup is called the *essential point symmetry group* of the equation \mathcal{F}'_V . For instance, for each of the potentials $V(x) = i\alpha x + \delta x^2$ with $\delta \in \{-1, 0, 1\}$ and $V(x) = i\alpha x + x$, the group G_V^{ess} is a five-dimensional Lie group with two components; its identity component $G_{V,\text{id}}^{\text{ess}}$ is singled out by the constraint $\varepsilon = 1$. The only independent (up to composing with elements of $G_{V,\text{id}}^{\text{ess}}$) discrete transformation in G_V^{ess} is the composition of the Wigner time reflection and the space reflection, $\tilde{t} = -t$, $\tilde{x} = -x$, $\tilde{\psi} = \psi^*$ for the potentials $V(x) = i\alpha x + \delta x^2$ with $\delta \in \{-1, 0, 1\}$ and the transformation $\tilde{t} = -t$, $\tilde{x} = -x$, $\tilde{\psi} = \exp(2it(x - t^2) + \frac{2}{3}\alpha t^3)\psi^*$ for $V(x) = i\alpha x + x$.

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- [1] Kurujyibwami C., Basarab-Horwath P. and Popovych R.O., Algebraic method for group classification of (1+1)-dimensional linear Schrödinger equations, *Acta Appl. Math.* **157** (2018), 171–203, arXiv:1607.04118.