Differentiable structures on line with n+1 origins

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I extended the arguments from [1] to obtain a classification of differentiable structure on the line with $n+1$ origins L_n up to a diffeomorphisms which fix Hausdorff-violating points.

Definition 1. Let τ be standard topology on $\mathbb{R}$. Let L_n be a disjoint union of $\mathbb{R}$ with the set $\{0_1, 0_2, \ldots, 0_n\}$ endowed with the following topology:

$$\eta = \tau \cup \coprod_{i=1}^{n} \big\{ (W \setminus \{0\}) \cup \{0_i\} : 0 \in W \in \tau \big\}$$

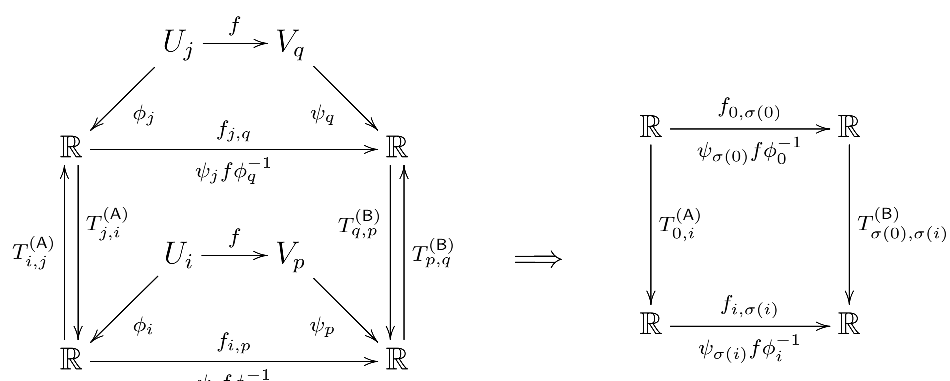
with additional elements being the open neighborhoods of 0 in which 0 is replaced by 0_i.

The space L_n is locally Euclidean and T_1, but not T_2 as the set $\mathrm{Hv} := \{0, 0_1, 0_2, \ldots, 0_n\}$ consists precisely of points for which the T_2 separation axiom fails to hold. Such points, are often called *Hausdorff-violating*.

Definition 2. Let A be a differentiable atlas on L_n. If A consists of charts of the form $\{(U_i, \phi_i)\}_{i=0}^{n}$ where $U_i := \mathbb{R} \setminus \{0\} \cup \{0_i\}$ ($U_0 := \mathbb{R}$), $\phi_i(0_i) = 0$ ($\phi_0(0) = 0$) and ϕ_i maps $(0; +\infty) \to (0; +\infty)$, then A is called the *minimal atlas* on L_n.

For any two charts in miminal atlas the transition map is a monotone $\mathcal{C}^r$-diffeomorphism of $\mathbb{R} \setminus \{0\}$. Therefore there exists a unique *continuous extension* of $\phi_j \phi_i^{-1}$ to the homeomorphism of $\mathbb{R}$ such that $0 \mapsto 0$. Let $T_{i,j}^{(\mathsf{A})}$ ($T_j^{(\mathsf{A})}$ if $i = 0$) denote such extension in some atlas A .

In general, if f is a $\mathcal{C}^r$-diffeomorphism which acts on Hv by a permutation σ, then we have the following commutative diagrams and corresponding relations between extensions of transition maps.



$$\begin{cases} f_{i,p} = T_{q,p}^{(\mathsf{B})} f_{j,q} T_{i,j}^{(\mathsf{A})} \\ T_{q,p}^{(\mathsf{B})} = f_{i,p} T_{j,i}^{(\mathsf{A})} f_{j,q}^{-1} \end{cases}$$

$$T_{\sigma(0),\sigma(i)}^{(\mathsf{B})} = f_{i,\sigma(i)} T_{0,i}^{(\mathsf{A})} f_{0,\sigma(0)}^{-1}$$

For each $r \in \mathbb{N} \cup \{\infty\}$ let

- $\mathcal{W}_{\mathbb{R}} := \mathcal{W}^r(\mathbb{R}, 0)$ be the group of homeomorphisms of $\mathbb{R}$ such that any element of $\mathcal{W}_{\mathbb{R}}$ maps 0 to itself and is a monotone $\mathcal{C}^r$-diffeomorphism of $\mathbb{R} \setminus \{0\}$;
- $\mathcal{D}_{\mathbb{R}} := \mathcal{D}^r(\mathbb{R}, 0)$ be the group of $\mathcal{C}^r$-diffeomorphisms of $\mathbb{R}$ which fix 0. Note that $\mathcal{D}_{\mathbb{R}}$ is a subgroup of $\mathcal{W}_{\mathbb{R}}$.

Proposition 1. *Let $\mathfrak{A}$ be some differentiable structure on L_n. Then $\mathfrak{A}$ contains a minimal atlas A as subatlas. Moreover, there is a one-to-one correspondence between minimal atlases and elements $\Xi^{(\cdot)} := (T_i^{(\cdot)})_{i=1}^n$ of the direct product $\prod_{i=1}^n \mathcal{W}_{\mathbb{R}}$.*

If f fix Hausdorff-violating points, the corresponding permutation σ is trivial and the theorem follows

Theorem 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two differentiable structures (maximal atlases) on L_n, then the following conditions are equivalent:*

1. *There exists a diffeomorphism $(L_n, \mathfrak{A}) \to (L_n, \mathfrak{B})$ which fix all Hausdorff-violating points;*
2. *For any pair of minimal atlases $\mathsf{A} \in \mathfrak{A}$ and $\mathsf{B} \in \mathfrak{B}$, elements $\Xi^{(\mathsf{A})}$ and $\Xi^{(\mathsf{B})}$ lie in a same orbit of action*

$$\mathcal{D}_{\mathbb{R}} \times \prod_{i=1}^n \mathcal{D}_{\mathbb{R}} \times \prod_{i=1}^n \mathcal{W}_{\mathbb{R}} \to \prod_{i=1}^n \mathcal{W}_{\mathbb{R}}$$
$$(f_0, f_1, \ldots, f_n).(T_1, \ldots, T_n) \mapsto (f_1 T_1 f_0^{-1}, \ldots, f_n T_n f_0^{-1})$$

The definition of $\mathcal{C}^r$-structure, $\mathcal{C}^r$-atlas and $\mathcal{C}^r$-diffeomorphism can be found in the book by J. Lee [3] and paper of F. Takens [2] with references therein.

[1] Lysynskyi M., Maksymenko S., Classification of differentiable structures on the non-Hausdorff line with two origins, (2024), 29 pp., arXiv:2406.09576.

[2] Takens F., Characterization of a differentiable structure by its group of diffeomorphisms, *Bol. Soc. Brasil. Mat.*, **10** (1979), no. 1, 17–25.

[3] Lee J. M., Introduction to smooth manifolds, volume 218 of Graduate Texts in Mathematics, Springer, New York, 2013, 726 pp.