On de Branges-Rovnyak spaces and their decompositions

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Let H^2 be the classical Hardy space on the disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$. For $\varphi \in L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_φ on H^2 is defined by $T_\varphi f = P(\varphi f)$, where P is the Szegö projection, i.e., the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2.

De Branges-Rovnyak spaces are a class of reproducing kernel Hilbert spaces determined by contractive analytic functions. More exactly, for each $b \in H^\infty$, $\|b\|_\infty \leq 1$, the de Branges-Rovnyak space $\mathcal{H}(b)$ is the image of H^2 under the operator $(1 - T_b T_{\overline{b}})^{1/2}$ with the range norm $\|\cdot\|_b$ defined by $\|(1 - T_b T_{\overline{b}})^{1/2} f\|_b = \|f\|$ for $f \in (\ker(1 - T_b T_{\overline{b}}))^\perp$, where $\|\cdot\|$ denotes the standard norm in H^2. The reproducing kernel is given by

$$k_w^b(z) = \frac{1 - \overline{b(w)} b(z)}{1 - \overline{w} z}, \quad z, w \in \mathbb{D}.$$

If b is an inner function, i.e., $|b| = 1$ a.e. on $\partial\mathbb{D}$, then $\mathcal{H}(b)$ is the well-known model space $K_b = H^2 \ominus bH^2$. A comprehensive treatment of the theory of the de Branges-Rovnyak spaces can be found in [7] and [4].

Here we focus on the case when b is a nonextreme point of the unit ball of H^∞, which is equivalent to $\log(1 - |b|) \in L^1(\partial\mathbb{D})$. In this case there exists an outer function $a \in H^\infty$ such that $|b|^2 + |a|^2 = 1$ a.e. on $\partial\mathbb{D}$. If we assume that $a(0) > 0$, then a is uniquely determined. It is known that for the spaces $\mathcal{M}(a) = aH^2$ and $\mathcal{M}(\overline{a}) = T_{\overline{a}} H^2$ with the corresponding range norms, the following contractive inclusions hold:

$$\mathcal{M}(a) \subset \mathcal{M}(\overline{a}) \subset \mathcal{H}(b).$$

If $\mathcal{M}(a)$ is not dense in $\mathcal{H}(b)$ one can ask what is its orthogonal complement.

A simple example is $b(z) = \frac{1+z}{2}$ with $a(z) = \frac{1-z}{2}$. D. Sarason [6] showed that

$$\mathcal{H}(\tfrac{1+z}{2}) = \mathcal{M}(\tfrac{1-z}{2}) \oplus_b \mathbb{C}$$

(note that $\mathbb{C}$ can be seen as the model space K_z). Sarason's result was generalized by several authors for $b = \frac{1+u}{2}$ with u a nonconstant inner function. Here $a = \frac{1-u}{2}$ and

$$\mathcal{H}(\tfrac{1+u}{2}) = \mathcal{M}(\tfrac{1-u}{2}) \oplus_b K_u$$

(see [1–3]).

In our talk we present the following generalizations of the above decomposition, obtained in joint work with M. Michalska and M. Nowak [5].

Theorem 1 ([5]). *Let u be a nonconstant inner function and let k be a nonnegative integer. Then*

$$\mathcal{H}(\tfrac{u^k(1+u)}{2}) = \mathcal{M}(\tfrac{1-u}{2}) \oplus_b (2 + 2u + \ldots + 2u^{k-1} + u^k) K_u.$$

Theorem 2. *Let u be a nonconstant inner function and let $k \geq 0$ be an odd integer. Then*

$$\mathcal{H}(\tfrac{u^k(1+u^2)}{2}) = \mathcal{M}(\tfrac{1-u^2}{2}) \oplus_b (1-u)q(u)K_u \oplus_b p(u)K_u,$$

where p and q are analytic polynomials defined by

$$p(z) = 2 + 2z + \ldots + 2z^{k-1} + z^k + z^{k+1}$$

and

$$q(z) = \tfrac{(k+1)p(-z)-p(z)}{1-z}.$$

For $k = 1$ Theorem 2 was established in [5]; the general case is new.

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