Invertibility of quantum channels induced by a family of projections

V. O. Kuzko[1], **V. I. Rabanovich**[2]

[1]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

[2]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

vkuzko@imath.kiev.ua, slavik@imath.kiev.ua

Let $M_n(\mathbb{C})$ be a space of $n \times n$ square matrices defined over complex field $\mathbb{C}$.

Consider a linear and completely positive map

$$\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \tag{1}$$

to be a quantum channel induced by a family of projections:

$$\Phi(X) = \sum_{i=1}^{s} \lambda_i P_i X P_i. \tag{2}$$

Following constraints must be satisfied:

1. $$P_i = P_i^* = P_i^2 \ \forall i \in \{1, .., s\}, \text{ where } P_i^* \text{ is conjugate transpose of } P_i. \tag{3}$$

2. $$\sum_{i=1}^{s} \lambda_i P_i = I \tag{4}$$

3. $$0 \le \lambda_i \le 1 \ \forall i \in \{1, .., s\} \tag{5}$$

Problem 1. The problem of existence of collections of projections that satisfy (3) for different λ_i was partially solved by S. A. Kruglyak in [1], and for equal coefficients λ_i there is a complete solution proposed in [2].

Our main goal is to find quantum channel of type (2) for which Φ is invertible. Another subsequent goal is to establish additional properties of eigenvalues of such channels.

Theorem 1. *Let $P_i \in M_n(\mathbb{C})$ and $k_i = rank(P_i), i = 1, 2, 3, ..., s$. Provided Φ is nonsingular, the following inequality holds:*

$$\sum_{i=1}^{s} k_i^2 \ge n^2$$

Thus, maps with small s and small $rank(P_i)$ are necessarily singular for n big enough.

Proposition 1. *For $s = 4$ we consider a quantum channel $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ of the form*

$$\Phi(X) = \frac{1}{2} \sum_{i=1}^{4} P_i X P_i$$

with $rank(P_i) = 1, i = 1, 2, 3, 4$*. It was shown that if every* P_i *has only real entries, then* Φ *is singular. And if only* $P_1, P_2 \in M_2(\mathbb{R})$ *but some entries of* P_3 *and* P_4 *have nonzero imaginary part, then* Φ *is invertible for all but one case*

$$P_1 \cdot P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

.

Proposition 2. *For* $s = 6$ *and* $\lambda_i = \frac{6}{13}, i = 1, 2, 3, 4, 5, 6$*, we took known construction of projections* $P_i \in M_6(\mathbb{C})$ *that satisfy* (3) *and found that nonsingular* Φ *has 11 eigenvalues with largest of them to be 1 of simple multiplicity. The spectral gap of* $\sigma(\Phi)$ *is* $\frac{1}{26}$*.*

Theorem 2. *Consider a case of Kraus decomposition*

$$\Phi(X) = \sum_{i=1}^{s} V_i^* X V_i,\ V_i \in M_n(\mathbb{C})$$

of unital quantum channel. Then, Φ *has a simple eigenvalue* $\alpha = 1$ *if and only if a collection* $V_1, V_2, ..., V_s$ *is irreducible.*

[1] S. A. Kruglyak, Coxeter functors for a certain class of *-quivers and *-algebras, *Methods Funct. Anal. Topology* **8** (2002), no. 4, 49–57.

[2] S. A. Kruglyak, V. I. Rabanovich, and Yu. S. Samoilenko. On sums of projections. *Functional Analysis and Its Applications*, **36** (2002), no. 3, 182–195.