# Approximation of stochastic differential equations with interaction based on the Monte Carlo method

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We consider the stochastic differential equation with interaction [1]

$$\begin{aligned} &\mathrm{d}x(u,t) = a\big(x(u,t),\mu_t\big)\,\mathrm{d}t + \int_{\mathbb{R}^d} b\big(x(u,t),\mu_t,p\big)\,W(\mathrm{d}p,\mathrm{d}t),\\ &x(u,0)=u, \qquad u\in\mathbb{R}^d, \quad t\in[0,1], \qquad \mu_t=\mu_0\circ x(\cdot,t)^{-1}. \end{aligned} \tag{1}$$

where

$$a:\mathbb{R}^d\times\mathfrak{M}_2(\mathbb{R}^d)\to\mathbb{R}^d, \quad b:\mathbb{R}^d\times\mathfrak{M}_2(\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}^{d\times d}.$$

Here $\mathfrak{M}_p(\mathbb{R}^d)$ is the set of all probability measures $\mu$ on the Borel $\sigma$-field on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d}|v|^p\,\mu(\mathrm{d}v)<+\infty, \qquad p\ge 1.$$

**Definition 1.** For $\mu,\nu\in\mathfrak{M}_p(\mathbb{R}^d)$, we denote by $\gamma_p(\mu,\nu)$ the Wasserstein distance of order $p$,

$$\gamma_p(\mu,\nu):=\left(\inf_{\kappa\in C(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,\kappa(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}}, \tag{2}$$

where $C(\mu,\nu)$ is the set of all couplings of $\mu$ and $\nu$.

We assume that the coefficients of (1) satisfy the Lipschitz condition with respect to the Euclidean norm in $\mathbb{R}^d$ and the Wasserstein distance $\gamma_2$. Under these assumptions, the stochastic differential equation with interaction admits a unique solution on $[0,+\infty)$. Moreover, for every $t>0$, the measure $\mu_t=\mu_0\circ x(\cdot,t)^{-1}$ is a random element in $\mathfrak{M}_2(\mathbb{R}^d)$ [1].

We define the Euler–Maruyama approximation associated with the Monte Carlo initial measure $\mu_0^N$ as follows. Let

$$t_k=\frac{k}{n}, \qquad k=0,1,\dots,n,$$

be the uniform partition of $[0,1]$. For $t\in[t_k,t_{k+1})$, set

$$\begin{aligned} x_n^N(t,u) &= x_n^N(t_k,u)+a\big(x_n^N(t_k,u),\mu_{t_k}^{N,n}\big)(t-t_k)\\ &\quad+\int_{t_k}^t\int_{\mathbb{R}^d} b\big(x_n^N(t_k,u),\mu_{t_k}^{N,n},p\big)\,W(\mathrm{d}p,\mathrm{d}s), \end{aligned} \tag{3}$$

with initial condition

$$x_n^N(0,u)=u,$$

where

$$\mu_{t_k}^{N,n}=\mu_0^N\circ\big(x_n^N(\cdot,t_k)\big)^{-1}.$$

To reduce the number of spatial nodes required by the discretization procedure and to avoid the dimensional dependence of regular grid constructions, we propose to approximate the initial measure by an empirical Monte Carlo measure

$$\mu_0^N=\frac{1}{N}\sum_{i=1}^N\delta_{U_i}, \qquad U_1,\dots,U_N\overset{\text{i.i.d.}}{\sim}\mu_0. \tag{4}$$

Assume that, for some $q > 2$, the initial measure $\mu_0$ has a finite $q$-th moment,

$$\int_{\mathbb{R}^d} |u|^q \, \mu_0(\mathrm{d}u) < +\infty.$$

**Lemma 1.** *There exists a constant $C > 0$, depending only on $d$, $q$, such that*

$$\mathbb{E}\, \gamma_2^2(\mu_0, \mu_0^N) \le C\, N^{-\frac{2(q-2)}{q(d+4)}}.$$

Let $p > d$, and choose

$$\gamma \in \left(0, \frac{2p-d}{2p}\right), \qquad \sigma := \frac{\gamma}{\gamma + d}.$$

For a compact set $K \subset \mathbb{R}^d$, let $\lambda_d(K)$ denote its $d$-dimensional Lebesgue measure. The constants $C_p > 0$, $C(d,p,\gamma) > 0$, and $C_K^{\mathrm{time}} > 0$ denote the constants arising respectively from the $p$-th moment estimate, the spatial continuity estimate, and the time discretization estimate on $K$.

**Theorem 1.** *Assume that the conditions considered above hold. Then*

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} |x(u,t) - x_n^N(u,t)| \le \widetilde{C}_K \left( C^{\frac{\sigma}{2}} N^{-\frac{\sigma(q-2)}{q(d+4)}} + n^{-\frac{1}{2}} \right),$$

*where*

$$\widetilde{C}_K := \frac{\gamma + d}{d^{1-\sigma} \gamma^{\sigma}} \left(\lambda_d(K) C_p^{\frac{1}{2p}}\right)^{\sigma} \left(C(d,p,\gamma)\, C_p\right)^{\frac{1-\sigma}{2p}} (\mathrm{diam}\, K)^{(1-\sigma)\left(1 - \frac{d}{2p} - \gamma\right)} + C_K^{\mathrm{time}}.$$

**Corollary 1.** *Let*

$$\alpha := \frac{\sigma(q-2)}{q(d+4)}.$$

*Choosing the number of Monte Carlo nodes as*

$$N = \left\lceil n^{\frac{1}{2\alpha}} \right\rceil,$$

*we obtain the balanced estimate*

$$\mathbb{E} \sup_{u \in K} \sup_{t \in [0,1]} |x(u,t) - x_n^N(u,t)| \le \widetilde{C}_K \left( C^{\frac{\sigma}{2}} + 1 \right) n^{-\frac{1}{2}}.$$

[1] Dorogovtsev A. A., Stochastic flows with interaction and measure-valued processes, *International Journal of Mathematics and Mathematical Sciences* **67** (2003), no. 1, 4265–4284.

[2] Kustareva K. M., Difference approximation for equations with interaction, *Theory Stoch. Process.* **29** (2025), no. 1, 52–64.