On the structure of some one-generated Leibniz rings

L.A. Kurdachenko, P.Ye. Minaiev, O.O. Pypka
Oles Honchar Dnipro National University, Dnipro, Ukraine
lkurdachenko@gmail.com, minaevp9595@gmail.com, sasha.pypka@gmail.com

A set L endowed with two binary operations $+$ and $[\,,\,]$ is called a *Leibniz ring* or, more precisely, a left Leibniz ring, if it satisfies the following conditions:

(i) $(L,+)$ is an abelian group;

(ii) $[a,b+c]=[a,b]+[a,c]$ and $[a+b,c]=[a,c]+[b,c]$;

(iii) $[a,[b,c]]=[[a,b],c]+[b,[a,c]]$,

for all $a,b,c\in L$.

For many algebraic structures, a natural starting point is the study of objects generated by a single element. In the case of Lie rings, one-generated rings have a rather simple structure. For Leibniz rings, however, the corresponding problem turns out to be considerably more subtle. We take the first steps toward understanding such Leibniz rings (see also [1]). It is also worth noting that, for the closely related structure of Leibniz algebras, an analogous problem was successfully solved by the authors; a detailed account can be found in the monograph [2].

Let $C=\mathbb{Z}/k\mathbb{Z}$, where $k=0$ or k is a positive integer. For each $n\in\mathbb{N}$, let C_n be isomorphic to C as an additive group, and put

$$A=\bigoplus_{n\in\mathbb{N}} C_n.$$

Thus, every element of A can be written as a finite sequence $x=(\alpha_n)_{n\in\mathbb{N}}$, where $\alpha_n\in C$.

Define an operation $[\,,\,]$ on A as follows. If $x=(\alpha_n)_{n\in\mathbb{N}}$ and $y=(\beta_n)_{n\in\mathbb{N}}$, then

$$[x,y]=(\gamma_n)_{n\in\mathbb{N}},$$

where

$$\gamma_1=0,\qquad \gamma_{n+1}=\alpha_1\beta_n\quad\text{for all } n\in\mathbb{N}.$$

A direct verification shows that this operation is biadditive and satisfies the Leibniz identity

$$[x,[y,z]]=[[x,y],z]+[y,[x,z]]$$

for all $x,y,z\in A$. Hence A is a Leibniz ring.

Let $a_j=(\delta_{jn})_{n\in\mathbb{N}}$, where δ_{jn} is the Kronecker delta. Then

$$[a_1,a_n]=a_{n+1}\qquad\text{for all } n\in\mathbb{N}.$$

Therefore, the ring A is generated, as a Leibniz ring, by the single element a_1.

Now put

$$A_j=\bigoplus_{n\geq j} C_n.$$

Then

$$A=A_1\geq A_2\geq\cdots\geq A_n\geq A_{n+1}\geq\cdots,\qquad \bigcap_{n\in\mathbb{N}} A_n=\langle 0\rangle.$$

This gives the lower central series of the Leibniz ring A. Moreover, each A_n is generated as a subring by the element a_n.

If $C = \mathbb{Z}$, the constructed Leibniz ring is denoted by $\mathbf{FL}(1, \mathbb{Z})$. If $C = \mathbb{Z}/k\mathbb{Z} = \mathbb{Z}_k$, where k is a positive integer, then it is denoted by $\mathbf{FL}(1, \mathbb{Z}_k)$.

Let $R = \mathbb{Z}[X]$ and consider the additive abelian group

$$\mathbf{FL}(\mathbb{Z}[X]) = \langle a \rangle \oplus R,$$

where $\langle a \rangle$ is an infinite cyclic group. Thus every element of $\mathbf{FL}(\mathbb{Z}[X])$ has the form

$$u = \alpha_0 a + f(X), \qquad f(X) = \lambda_0 + \lambda_1 X + \cdots + \lambda_n X^n,$$

where $\alpha_0, \lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{Z}$.

For

$$u = \alpha_0 a + f(X), \qquad v = \beta_0 a + g(X),$$

where

$$g(X) = \mu_0 + \mu_1 X + \cdots + \mu_k X^k,$$

define the product $[\,,\,]$ by

$$[u, v] = \alpha_0\beta_0 + \alpha_0\mu_0 X + \alpha_0\mu_1 X^2 + \cdots + \alpha_0\mu_k X^{k+1}.$$

A direct computation shows that, for arbitrary elements $u, v, w \in \mathbf{FL}(\mathbb{Z}[X])$, one has

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]].$$

This construction gives a polynomial realization of the one-generated Leibniz ring. In particular, we have the following result.

Theorem 1. *The Leibniz rings $\mathbf{FL}(1, \mathbb{Z})$ and $\mathbf{FL}(\mathbb{Z}[X])$ are isomorphic.*

The next theorem shows that, in the torsion-free case, the above construction is essentially the only possible infinite-rank example among one-generated Leibniz rings.

Theorem 2. *Let L be a Leibniz ring whose additive group is torsion-free. Then either L is isomorphic to $\mathbf{FL}(1, \mathbb{Z})$, or L has finite 0-rank.*

The following theorem gives the corresponding analogue for Leibniz rings whose additive group is an elementary abelian p-group.

Theorem 3. *Let L be a Leibniz ring whose additive group is an elementary abelian p-group for some prime p. Then either L is isomorphic to $\mathbf{FL}(1, \mathbb{Z}_p)$, or L is finite.*

[1] Kurdachenko L.A., Pypka O.O., Semko M.M., On the structure of some Leibniz rings, Algebra Discrete Math. **40** (2025), 1, 79–108.

[2] Kurdachenko L.A., Pypka O.O., Subbotin I.Ya., General Theory of Leibniz Algebras, Springer, Switzerland, 2024, 166 pp.