Potential analysis of double phase heterogeneous models

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We develop qualitative methods for the study of nonlinear heterogeneous structures, models of which are elliptic equations, which describe complex nonlinear processes in heterogeneous media. They may also include the structures consisting of several parts (phases or layers): multiphase solid and liquid materials; optic fiber and optic cable layers; anisotropic medium, etc. Relevance of the chosen direction is due to the fact that many processes in heterogeneous environments under conditions of high temperatures, heavy loads and significant deformations are described using nonlinear differential equations with discontinuous (singular) data (coefficients, right-hand side, boundary and initial conditions, etc.). At the same time, the concept of weak solutions that meet the modern needs of mathematical physics arose. Nonlinear differential equations have a complex structure, which makes them impossible to study by finding solutions in an explicit form. Therefore, the development of qualitative methods for their investigations becomes an extremely important tool. Using potential theory [2], the behavior of a weak solution of this equation at a fixed point is estimated and analyzed by the value of the nonlinear Wolff potential from the right hand side. We study pointwise properties that play a key role in the further study: expansion of positivity Harnack's inequalities, regularities and others.

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider a double-phase elliptic equation with variable exponents:

$$-\mathrm{div} A(x,\, \nabla u) := -\mathrm{div}\left[(|\nabla u|^{p(x)-2} + a(x)|\nabla u|^{q(x)-2})\nabla u\right] = f(x) \geq 0, \tag{1}$$

where $f(x) \in L^1(\Omega)$. We assume that the function $A(x,\, \xi) = |\xi|^{p(x)-1} + a(x)|\xi|^{q(x)-1} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions

1) $A(x,\, \xi)$ satisfies the Carathéodory condition,

2) $A(x, \xi)\xi \geq \mu_1(|\xi|^{p(x)} + a(x)|\xi|^{q(x)})$,

3) $|A(x, \xi)| \leq \mu_2(|\xi|^{p(x)-1} + a(x)|\xi|^{q(x)-1})$ with some constants $\mu_1,\, \mu_2 > 0$.

4) We also assume that $0 \leq a(x) \in C^{0,\,\alpha}(\Omega)$, $\quad \alpha \in (0,\, 1]$. Let $\mathcal{M}$ be a set of all measurable functions, $p(x), q(x) : \Omega \to (1, \infty)$. For $p(x), q(x) \in \mathcal{M}$, we set: $p_- := \mathrm{essinf}_{x\in\Omega} p(x)$, $q_- = \mathrm{essinf}_{x\in\Omega} q(x)$, $p_+ := \mathrm{esssup}_{x\in\Omega} p(x)$, $q_+ = \mathrm{esssup}_{x\in\Omega} q(x)$, and $1 < p_- \leq p_+ \leq q_- \leq q_+ \leq \min\left(p_- + \alpha,\, \frac{n(p_- - 1)}{n - p_-}\right)$, $\quad q_+ < n$.

Let us introduce the necessary definitions.

Definition 1. [1–3] Let $G(x, t) = t(t^{p(x)-1} + a(x)t^{q(x)-1})$. Then $W^{1,G}(\Omega)$ denotes the class of functions u that are weakly differentiable in Ω and satisfy $\int\limits_{\Omega} G(a(x),\, |\nabla u|)\, dx < \infty$.

We will prove the pointwise estimates for a nonnegative weak solution to the double-phase equation (1) in terms of the nonlinear Wolff potentials, [1, 2]: $W^{f}_{1,p(x)}(x_0,\, R) = \sum_{j=0}^{\infty} \left(\rho_j^{p(x)-n} \int\limits_{B_{\rho_j}(x_0)} f\, dx\right)^{\frac{1}{p(x)-1}}$, $\rho_j = \frac{R}{2^j}$, $j = 0,\, 1,\, \ldots$ under the assumption that the series in the above formulae are convergent, i.e. the Wolff potentials are finite.

The main result of the present work is the following theorem.

Theorem 1. *[1] Let* $u \in W^{1,G}(\Omega) \cap L^\infty$ *be a nonnegative weak solution to Eq. (1). Let conditions 1)-4) above be satisfied, and let* $[a]_{C^{0,\alpha}(\Omega)} := \sup\limits_{x,y\in\Omega,\, x\neq y} \frac{|a(x)-a(y)|}{|x-y|^\alpha}$. *Assume also that the point* $x_0 \in \Omega$ *is such that* $B_{4\rho}(x_0) \subset \Omega$. *Then there exist constants* c_1, $c_2 > 0$ *depending only on* p_-, q_+, n, $[a]_{C^{0,\alpha}(\Omega)}$ *and* $||u||^{q_+-p_-}_{L^\infty(\Omega)}$ *such that, under condition* $a(x_0) = 0$ *the following estimate holds:*

$$c_1 W^f_{1,p_-}(x_0,\rho) \leq u(x_0) \leq c_2 \inf_{B_\rho(x_0)} u + c_2 W^f_{1,p_-}(x_0, 2\rho). \tag{2}$$

If $a(x_0) > 0$ *and* $\rho_0^\alpha = \frac{a(x_0)}{4[a]_{C^{0,\alpha}(\Omega)}} \geq \rho^\alpha$, *then there exist constants* c_3, $c_4 > 0$ *depending on* p_-, q_+, n, $[a]_{C^{0,\alpha}(\Omega)}$, $||u||^{q_+-p_-}_{L^\infty(\Omega)}$ *and* $a(x_0)$ *such that the following estimate*

$$c_3 W^f_{1,q_+}(x_0,\rho) \leq \rho + u(x_0) \leq 3\rho + c_4 \inf_{B_\rho(x_0)} u + c_4 W^f_{1,q_+}(x_0, 2\rho) \tag{3}$$

holds.

Under conditions $a(x_0) > 0$ *and* $\rho_0 < \rho$ *will be true the estimate*

$$c_3 W^f_{1,q_+}(x_0,\rho) + c_3(W^f_{1,p_-}(x_0,\rho) - W^f_{1,p_-}(x_0,\rho_0)) \leq \rho + u(x_0) \leq$$

$$\leq 3\rho + c_4 \inf_{B_\rho(x_0)} u + c_4 W^f_{1,q_+}(x_0,2\rho) + c_4(W^f_{1,p_-}(x_0,2\rho) - W^f_{1,p_-}(x_0,2\rho_0)). \tag{4}$$

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