A sinc-quadrature method for Neumann problem for Laplace equation in a planar domain using single-layer potential

Y.A. Kozlovskyi[1], **V.H.Vavrychuk** [1]

[1] Ivan Franko National University of Lviv

yurii.kozlovskyi@lnu.edu.ua, vasyl.vavrychuk@lnu.edu.ua

The Neumann problem for the Laplace equation serves as a mathematical model for a wide range of physical processes, including heat conduction, electrostatics, and fluid dynamics. Let $D \subset \mathbb{R}^2$ be an open simply connected bounded domain with Lipschitz boundary ∂D. We consider a problem of finding a function $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial D, \\ u(x) = o(1) & \text{for } |x| \to \infty, \end{cases} \tag{1}$$

where $g \in C(\partial D)$ and ν denotes the outward unit normal to D. It has a unique solution if and only if the condition

$$\int_{\partial D} g \, ds = 0 \tag{2}$$

is satisfied [1].

To reduce the problem (1) to a boundary integral equation, its solution could be represented via a single-layer potential. By applying the Neumann boundary condition and using the jump relation for the normal derivative of the single-layer potential, we obtain an integral equation

$$\varphi(x) - 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)} \varphi(y) ds(y) = -2g(x), \quad x \in \partial D, \tag{3}$$

where $\varphi \in C(\partial D)$ is an unknown density and $\Phi(x,y)$ is given by

$$\Phi(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}, \quad x \neq y, \tag{4}$$

and an additional condition to satisfy solution u required behaviour at the infinity is

$$\int_{\partial D} \varphi ds = 0. \tag{5}$$

Thus, we parametrize the boundary ∂D by a smooth function $x : [0,1] \to \partial D$ such that

$$\partial D = \{x(t) = (x_1(t), x_2(t)) : 0 \le t \le 1\}, \quad |x'(t)| > 0. \tag{6}$$

Under this parametrization, equation (3) is transformed into

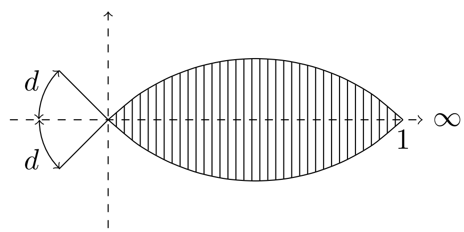
$$\mu(t) + \int_0^1 H(t,\tau)\, \mu(\tau)\, d\tau = \tilde{g}(t), \quad t \in [0,1], \tag{7}$$

where $\mu(t) = \varphi(x(t))$, $\tilde{g}(t) = -2g(x(t))$, and the kernel $H(t,\tau)$ is given by

$$H(t,\tau) = \begin{cases} \dfrac{1}{\pi} \dfrac{(x(t) - x(\tau)) \cdot \nu(x(t))}{|x(t) - x(\tau)|^2} \, |x'(\tau)|, & t \neq \tau, \\ \dfrac{1}{2\pi} \dfrac{x''(t) \cdot \nu(x(t))}{|x'(t)|}, & t = \tau. \end{cases} \tag{8}$$

Since equation (7) is a Fredholm integral equation of the second kind with a continuous kernel $H(t,\tau)$, we apply the Nyström method based on sinc-quadrature for its numerical solution. Sinc approximation methods are especially effective for problems whose solution may involve various types of singularities or for which other traditional methods do not achieve satisfactory accuracy. They allow to achieve exponential convergence rates under the appropriate conditions [3].

Definition 1. For $\alpha > 0$ and $0 < d < \pi/2$, the sinc function space $S_{\alpha,d}$ consists of all functions g real valued on $(0,1)$ and analytic in $E_d = \{w(z) = \dfrac{1}{1+e^{-z}},\ z \in \mathbb{C} : |\operatorname{Im}(z)| < d\}$



and satisfying the condition

$$\left|t(1-t)^{1-\alpha}\right|\left|g(t)\right| \le C, \quad t \in E_d \tag{9}$$

for some constant $C > 0$. The norm on $S_{\alpha,d}$ is defined by

$$\|g\|_{S_{\alpha,d}} = \sup_{t\in E_d} \left|t(1-t)^{1-\alpha}\right|\left|g(t)\right|, \quad t \in E_d. \tag{10}$$

Theorem 1. *Let $g \in S_{\alpha,d}$ with $\alpha > 0$ and $0 < d < \pi/2$. Then, for some $\lambda > 0$ and the step width $h = \lambda/n^{1/2}$, $n \in \mathbb{N}$ the quadrature error satisfies*

$$\Big|\int_0^1 g(t)\,dt - h\sum_{j=-n}^{n} w'(jh)g(w(jh))\Big| \le Ce^{-\sigma n^{1/2}}\|g\|_{S_{\alpha,d}}, \tag{11}$$

where $C > 0$ and $\sigma > 0$ are constants depending on d, α, and λ.

The application of a sinc-quadrature method within the Nyström method to solve the boundary integral equation of the second kind was investigated in the work of Kres, Sloan and Stenger [2]. They considered the Dirichlet problem for the Laplace equation in planar domains with corners and demonstrated the advantages of using such an approach.

The aim of this paper is to extend a sinc-quadrature method for solving the Neumann problem (1). Apart from domains with smooth boundaries, we could also consider the problem in planar domains with corners and evaluate the accuracy of the results.

[1] Kress R. Linear Integral Equations, 3rd. ed, Springer, New York, 2014, 412 pp.

[2] Kress R., Sloan I.H., Stenger F, A sinc quadrature method for the double-layer integral equation in planar domains with corners, *The Journal of Integral Equations and Applications* **10** (1998), 3, 291–317.

[3] Stenger F. Handbook of sinc numerical methods, CRC Press, London, 2011, 460 pp.