

GRAPH-ADMISSIBLE LIE ALGEBRAS

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A finite-dimensional Lie algebra \mathfrak{g} is called graph-admissible if it has a spanning set $\{e_1, \dots, e_m\}$ such that for all $i, j \in \{1, \dots, m\}$, there exists $k \in \{1, \dots, m\}$ and $\alpha \in \mathbb{F}$ with

$$[e_i, e_j] = \alpha \cdot e_k.$$

A finite-dimensional Lie algebra \mathfrak{g} is called minimal-graph-admissible if it has a basis with the mentioned property.

Let \mathfrak{g} be a graph-admissible Lie algebra with the corresponding spanning set $\{e_1, \dots, e_m\}$. Consider a directed graph with the vertex set $\{e_1, \dots, e_m\}$ and weighted arcs of the form $e_i \rightarrow e_k$ with label e_j for each bracket $[e_i, e_j] = \alpha \cdot e_k$ for non-zero scalars $\alpha \in \mathbb{F}$. This graph is said to be associated with \mathfrak{g} . We say a graph $G(V, E)$, associated with a Lie algebra \mathfrak{g} , is a minimal graph if $|V| = \dim(\mathfrak{g})$.

These notions were introduced and thoroughly studied in [1]. It is important to note that this approach continues the already established rich interplay between graphs and Lie algebras (we only mention the works [2, 3] with similar motives).

Using this construction, one can obtain graph-theoretic criteria for solvability, nilpotency, presence of ideals, simplicity, semisimplicity, and reductiveness of graph-admissible Lie algebras. In particular, it is proved that \mathfrak{g} is nilpotent if and only if the corresponding graph $G(V, E)$ is acyclic (see [1, Theorem 70]). Similarly, \mathfrak{g} is non-solvable if and only if $G(V, E)$ contains a strongly connected self-contained subgraph (see [1, Theorem 62]). Here, a subgraph H of $G(V, E)$ is called self-contained provided for every its arc, H contains a vertex with the label of this arc.

Remark. We note that the given proof of [1, Theorem 62] is rather lengthy (relying on a separate lemma) and can be substantially shortened by using the notion of a condensation of directed graphs. Namely, the condensation of a directed graph is the factor-graph over all strongly connected components. The idea for the shorter proof of necessity in [1, Theorem 62] is the following: since the condensation of $G(V, E)$ is always acyclic, consider any source in it – this vertex would correspond to a strongly connected component H in $G(V, E)$ which would be self-contained by construction. The proof of sufficiency in [1, Theorem 62] is simpler: any strongly connected subgraph H has a spanning closed walk implying that it does not have sources; additionally, if H is self-contained, it would be equal to its derived algebra (implying that its derived series will never reach zero).

Now we concentrate on two specific conjectures from [1], obtaining a proof of the first one in very specific case and disproving the second one.

Conjecture 1. [1, Conjecture 52] Let \mathfrak{g} be a finite-dimensional minimal-graph-admissible Lie algebra with non-trivial center. Then \mathfrak{g} admits a minimal graph such that the set of its sinks spans the center of \mathfrak{g} .

Theorem. Conjecture 1 holds for 2-nilpotent Lie algebras.

Conjecture 2. [1, Conjecture 80] Let \mathfrak{g} be a finite-dimensional minimal-graph-admissible Lie algebra associated with the minimal graph $G(V, E)$. If \mathfrak{g} has a non-zero center, then there

exists a proper non-empty subset $W \subset V$ that is closed in $G(V, E)$ and spans a solvable ideal in \mathfrak{g} .

In [1, page 47] it was established that Conjecture 2 holds for all 3-dimensional Lie algebras. However, it turns out that this conjecture is false for higher dimensions. To see this, consider a 4-dimensional Lie algebra \mathfrak{g} with basis $\{e_1, e_2, e_3, e_4\}$ and commutation relations

$$[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2, [e_4, e_2] = e_3, [e_3, e_4] = e_2.$$

It can be observed that $e_1 - e_4$ is a non-zero central element of \mathfrak{g} (in fact, \mathfrak{g} is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{g}_1$). Further, let $G(V, E)$ be the minimal graph of \mathfrak{g} associated with this basis. We see that the only non-empty proper closed subset of $G(V, E)$ is $W = \{e_1, e_2, e_3\}$. However, W spans an ideal isomorphic to $\mathfrak{so}(3)$, which is non-solvable.

The first author acknowledges a financial support of the National Research Foundation of Ukraine (the project number 2025.07/0405).

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