Conservation laws of multi-layer quasi-geostrophic problem

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The multi-layer quasi-geostrophic problem describes the dynamics of a stratified incompressible fluid with m layers of constant densities stacked in order of increasing density under the "rigid-lid" and flat-bottom approximations, see [1] and references therein for more details.

The quasi-geostrophic approximation assumes that the vertical component of the velocity field is negligibly small, while the horizontal motion satisfies a system of m ($m \geqslant 2$) coupled vorticity equations. These are third-order partial differential equations of the form

$$\begin{aligned} &q^i_t + \{\psi^i, q^i\} = 0, \\ &q^i := \psi^i_{xx} + \psi^i_{yy} + f_{i,i-1}(\psi^{i-1} - \psi^i) - f_{i,i+1}(\psi^i - \psi^{i+1}) + \beta y, \quad i = 1, \dots, m. \end{aligned} \tag{1}$$

Here $\psi^i = \psi^i(t,x,y)$ and $q^i = q^i(t,x,y)$ are the stream function and the quasi-geostrophic potential vorticity for the ith layer, respectively. Their Poisson bracket is given by the Jacobian with respect to (x,y), $\{\psi^i, q^i\} := \psi^i_x q^i_y - \psi^i_y q^i_x$. The real parameters $f_{i,i-1}$ and $f_{i,i+1}$, with the boundary conventions $f_{1,0}, f_{m,m+1} := 0$, describe the vertical coupling between adjacent layers.

A more convenient representation of the system (1), particularly for the study of conservation laws, is that in the matrix form:

$$\begin{aligned} &q_t + \{\psi, q\} = 0, \\ &q := \psi_{xx} + \psi_{yy} + \mathsf{F}\psi + \beta y \bar{1}, \end{aligned} \tag{2}$$

where $\psi := (\psi^1, \dots, \psi^m)^\mathsf{T}$, $q := (q^1, \dots, q^m)^\mathsf{T}$, the Poisson bracket $\{\psi, q\}$ is defined componentwise, $\{\psi, q\} := (\{\psi^i, q^i\})^\mathsf{T} = \psi_x \odot q_y - \psi_y \odot q_x$, "$\odot$" is the Hadamard (componentwise) product of matrices, the vertical coupling matrix $\mathsf{F} := (f_{ij})$ is a constant m-by-m tridiagonal matrix with $f_{ii} := -f_{i,i-1} - f_{i,i+1}$, $f_{10} := 0$, $f_{m,m+1} := 0$ and $f_{ij} := 0$ if $|i-j| > 1$, and $\bar{1}$ denotes the all-ones m-column, $\bar{1} := (1, \dots, 1)^\mathsf{T}$. We refer to the off-diagonal components $f_{i+1,i}$ and $f_{i,i+1}$, $i = 1, \dots, m-1$, of the matrix F as the essential ones.

Investigating Hamiltonian structures and conservation laws of (1) requires understanding some rather non-trivial spectral properties of the matrix F. It turns out that under the physical assumption of positivity of the essential components of F, the matrix F is diagonalizable, it has negative real pairwise different eigenvalues, and has rank $m-1$. These spectral properties naturally lead to endowing the space $\mathbb{R}^m$ with a weighted inner product $(\cdot,\cdot)_\mathsf{W}$ defined by the weight matrix $\mathsf{W} := \operatorname{diag}(b_1, \dots, b_m)$, where $b_1 := 1$ and

$$b_i := \frac{f_{12} \cdots f_{i-1,i}}{f_{21} \cdots f_{i,i-1}}, \quad i = 2, \dots, m.$$

In this Euclidean space, the eigenvectors of F are orthogonal.

Below we list the names of conservation laws of (2) alongside their canonical characteristics, canonical forms of their conserved currents, which are usually not unique, and the corresponding conserved quantities. Let ρ_0 denote the reference density of the fluid and H_i denote the rest height of the ith layer.

Conservation of generalized total weighted circulations. $\lambda=\kappa(t)\mathsf{W}\bar{1}$,

$$\big(\kappa(\bar{1},\triangle\psi)_{\mathsf{W}},\ -\kappa_t(\bar{1},\psi_x)_{\mathsf{W}}-\kappa(\psi_y,q)_{\mathsf{W}},\ -\kappa_t(\bar{1},\psi_y)_{\mathsf{W}}+\kappa(\psi_x,q)_{\mathsf{W}}\big).$$

Multiplying this conserved current by $H_1/(H_1+\cdots+H_m)$ yields a representation for the conservation of the generalized total weighted circulation, which is the weighted average of the generalized total circulations over the individual layers,

$$\sum_{j=1}^m\frac{H_j}{H_1+\cdots+H_m}\iint\kappa\triangle\psi^j\,\mathrm{d}x\mathrm{d}y.$$

Conservation of generalized total zonal momentums. $\lambda=-\chi(t)y\mathsf{W}\bar{1}$,

$$\big(\chi(\bar{1},\psi_y)_{\mathsf{W}},\ \chi\big(y(\psi_y,q)_{\mathsf{W}}-\tfrac12|\psi_x|^2_{\mathsf{W}}-\tfrac12|\psi_y|^2_{\mathsf{W}}-\tfrac12(\psi,\mathsf{F}\psi)_{\mathsf{W}}-y(\bar{1},\psi_{tx}+\beta\psi)_{\mathsf{W}}\big),$$
$$-\chi(\psi_x,yq-\psi_y)_{\mathsf{W}}-(\bar{1},\chi y\psi_{ty}+\chi_t\psi)_{\mathsf{W}}\big).$$

We multiply this conserved current by ρ_0H_1. The first component of the obtained tuple is the density of the generalized total zonal momentum, which equals the sum of the total zonal momenta on single layers,

$$\sum_{i=1}^m\iint\chi\rho_0H_i\psi^i_y\,\mathrm{d}x\mathrm{d}y.$$

Conservation of total energy. $\lambda=-\mathsf{W}\psi$,

$$\big(\tfrac12(|\psi_x|^2_{\mathsf{W}}+|\psi_y|^2_{\mathsf{W}}-(\psi,\mathsf{F}\psi)_{\mathsf{W}}),\ -(\psi,\psi_{tx}+\tfrac12\psi\odot q_y)_{\mathsf{W}},\ -(\psi,\psi_{ty}-\tfrac12\psi\odot q_x)_{\mathsf{W}}\big).$$

After multiplying this conserved current by ρ_0H_1, its first component yields total energy density. The corresponding conserved energy functional reads as

$$\mathcal{E}:=\sum_{i=1}^m\iint\frac{\rho_0H_i}{2}\big((\nabla\psi^i)^2-\psi^i(\mathsf{F}\psi)^i\big)\mathrm{d}x\mathrm{d}y.$$

Conservation of generalized potential enstrophies on the ith layer, $i\in\{1,\dots,m\}$.

$$\lambda=\frac{\mathrm{d}\Phi^i}{\mathrm{d}q^i}\delta_i,\quad (\Phi^i,-\psi^i_y\Phi^i,\psi^i_x\Phi^i),$$

where $\delta_i=(\delta_{ij},j=1,\dots,m)^\mathsf{T}$ with the Kronecker delta δ_{ij}, Φ^i is an arbitrary smooth function of q^i. For each fixed i, specific choice of Φ^i yield familiar physical invariants within the family of conserved functionals

$$\mathcal{C}_{i,\Phi^i}:=\iint\Phi^i(q^i)\mathrm{d}x\mathrm{d}y.$$

For example, the conservation of total potential circulation on the ith layer when $\Phi^i=q^i$, and the conservation of potential enstrophy on the ith layer when $\Phi^i=\frac12(q^i)^2$.

[1] Koval S., Bihlo A. and Popovych R., Symmetry analysis and exact solutions of multi-layer quasi-geostrophic problem, 2026, 65 pp., arXiv:2603.26959.