Mixed Volume Inequalities for Zonoids

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1. Mixed volumes and zonoids. For convex bodies $K_1, \dots, K_n \subset \mathbb{R}^n$, the *mixed volume* $V(K_1, \dots, K_n)$ is defined by the polarisation identity

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{j=1}^{n} (-1)^{n+j} \sum_{1 \le i_1 < \cdots < i_j \le n} \mathrm{vol}(K_{i_1} + \cdots + K_{i_j}),$$

where the sums are Minkowski sums. Mixed volumes are symmetric, multilinear with respect to Minkowski addition, and satisfy $V(K, \dots, K) = \mathrm{vol}(K)$. They play a central role in the Brunn-Minkowski theory and appear in inequalities such as the Alexandrov-Fenchel inequality.

A *zonotope* is the Minkowski sum of finitely many line segments, or equivalently, it is a linear projection of a cube. A *zonoid* is a limit of zonotopes in the Hausdorff metric. Zonoids form a rich class that includes all centrally symmetric convex bodies in $\mathbb{R}^2$ but not in higher dimensions. They are interesting for several reasons: they admit a simple analytic description via support functions $h_Z(u) = \int_{S^{n-1}} |\langle u, v \rangle| \, d\mu(v)$. Also volume calculations for zonoids reduce to determinants, linking geometry with linear algebra. Observe that in statistics, the Vitale zonoid $Z(X) = \mathbb{E}[0, X]$ of a random vector X provides a geometric summary of dependence structure. Many inequalities that hold for all convex bodies become sharp or take a simpler form when restricted to zonoids.

2. Log-submodularity conjecture. Fradelizi, Madiman, Meyer and Zvavitch proposed the following inequality for zonoids: for any zonoid A and any finite collection of zonoids $B_1, \dots, B_k$,

$$\mathrm{vol}(A) \cdot \mathrm{vol}(A + B_1 + \cdots + B_k) \le \prod_{i=1}^{k} \mathrm{vol}(A + B_i). \tag{1}$$

This is a multiplicative diminishing-returns principle. Taking logarithms shows it is a form of submodularity of $\log \mathrm{vol}$ with respect to Minkowski addition. The conjecture was proved for $n = 3$ in [1]. The present work establishes the next case $n = 4$.

Using mixed volumes, (1) is equivalent to *Bezout type inequality*: for $1 \le k \le n$,

$$\mathrm{vol}(A)^{k-1} V(A[n-k], B_1, \dots, B_k) \le \frac{n^k (n-k)!}{n!} \prod_{i=1}^{k} V(A[n-1], B_i). \tag{2}$$

Further reduction via orthogonal projections leads to a purely combinatorial inequality on the d-dimensional cube, $d = n - 1$, called the *hypercube inequality.* For $n = 4$ we have $d = 3$, and the hypercube inequality takes the following concrete form.

3. Main result: hypercube inequality for $d = 3$. Let $s_1, \dots, s_8$ be non-negative real numbers assigned to the vertices of the unit cube $[0, 1]^3$ (labelled in the natural order).

Theorem 1 (Hypercube inequality for $d = 3$)**.**

$$\Big(\sum_{i=1}^{8} s_i\Big)^2 \Bigg[\Big(\sum_{i=1}^{4} s_i\Big)\Big(\sum_{5\le i<j<k\le 8} s_i s_j s_k\Big) + \Big(\sum_{i=5}^{8} s_i\Big)\Big(\sum_{1\le i<j<k\le 4} s_i s_j s_k\Big)$$
$$+ (s_1+s_2)(s_3+s_4)(s_5s_6+s_7s_8) + (s_1+s_3)(s_2+s_4)(s_5s_7+s_6s_8)$$
$$+ (s_1+s_4)(s_2+s_3)(s_6s_7+s_5s_8) + 2s_1s_2s_6s_7 + 2s_2s_3s_5s_8\Bigg]$$
$$\le \Big(\sum_{i=1}^{4} s_i\Big)\Big(\sum_{i=5}^{8} s_i\Big)\Big(\sum_{i\in\{1,3,5,7\}} s_i\Big)\Big(\sum_{i\in\{2,4,6,8\}} s_i\Big)\Big(\sum_{i\in\{1,2,5,6\}} s_i\Big)\Big(\sum_{i\in\{3,4,7,8\}} s_i\Big).$$

Equality holds if and only if at least one of the six factors on the right-hand side is zero.

Theorem 1 immediately implies Bezout type inequality (2) for $n = 4$, and therefore the log-submodularity conjecture (1) for zonoids in $\mathbb{R}^4$. Moreover, the projection inequality

$$\operatorname{vol}(A)^{k-1}\operatorname{vol}(P_{[b_1,\dots,b_k]^\perp}A) \le \prod_{i=1}^{k}\operatorname{vol}(P_{b_i^\perp}A)$$

is verified for any orthonormal basis $\{b_1,\dots,b_4\}$ and any zonoid $A\subset\mathbb{R}^4$. In other words, the volume of a zonoid projected onto a subspace of codimension k is bounded by the product of the volumes of its projections onto the k corresponding coordinate hyperplanes, revealing a precise geometric control of high-dimensional volume through lower-dimensional data.

[1] Fradelizi M., Madiman M.,,Meyer M., Zvavitch A., On the log-submodularity of the volume of zonoids, *Commun. Contemp. Math.* (2023).

[2] Averkov G., von Dichter K., Soprunov I., Mixed volume inequalities for zonoids, Preprint (2026).