AN APERIODIC TILING BASED ON INTERIOR PROPERTIES OF SELF-SIMILAR SETS

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Objects with complex local structures, including fractals, are frequently defined as sets of parameter values for which a dynamical system satisfies or fails to satisfy a certain property. In particular, the Mandelbrot set can be treated as the locus of points $c \in \mathbb{C}$ such that the filled Julia set, associated with the holomorphic mapping $f(z) = z^2 + c$, is connected. Another famous example is the Rauzy gasket – the set of parameters on a two-dimensional simplex for which the Rauzy induction is defined for all iterations and the corresponding sequence of Rauzy moves is infinite and not eventually periodic or reducible.

On the real line, we consider a parameterised family of self-similar iterated function systems

$$\mathcal{F}_{x,y} = \left\{ f_1(\alpha) = \frac{\alpha}{4}, f_2(\alpha) = \frac{\alpha + x}{4}, f_3(\alpha) = \frac{\alpha + y}{4}, f_4(\alpha) = \frac{\alpha + x + y}{4} \right\},$$

where $(x, y) \in \mathbb{N}^2$. The attractor of $\mathcal{F}_{x,y}$ is the unique compact set $E(x, y) \subset \mathbb{R}$ that satisfies

$$E(x, y) = \bigcup_{f \in \mathcal{F}_{x,y}} f(E(x, y)).$$

We are interested in determining pairs (x, y) for which the attractor has a non-empty or empty interior.

We introduce a sequence of positive integers V for which the last non-zero digit of the quaternary expansion is either 1 or 3. Consequently, its complement D comprises all numbers for which the rightmost non-zero digit in the quaternary expansion equals 2. Taking this definition into account, we can write the first terms of the sequences:

$$V = \{1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 25, 27, 28, 29, 31, 33, \dots\}$$

and

$$D = \{2, 6, 8, 10, 14, 18, 22, 24, 26, 30, 32, 34, 38, 40, 42, 46, 50, 54, 56, 58, 62, \dots\}.$$

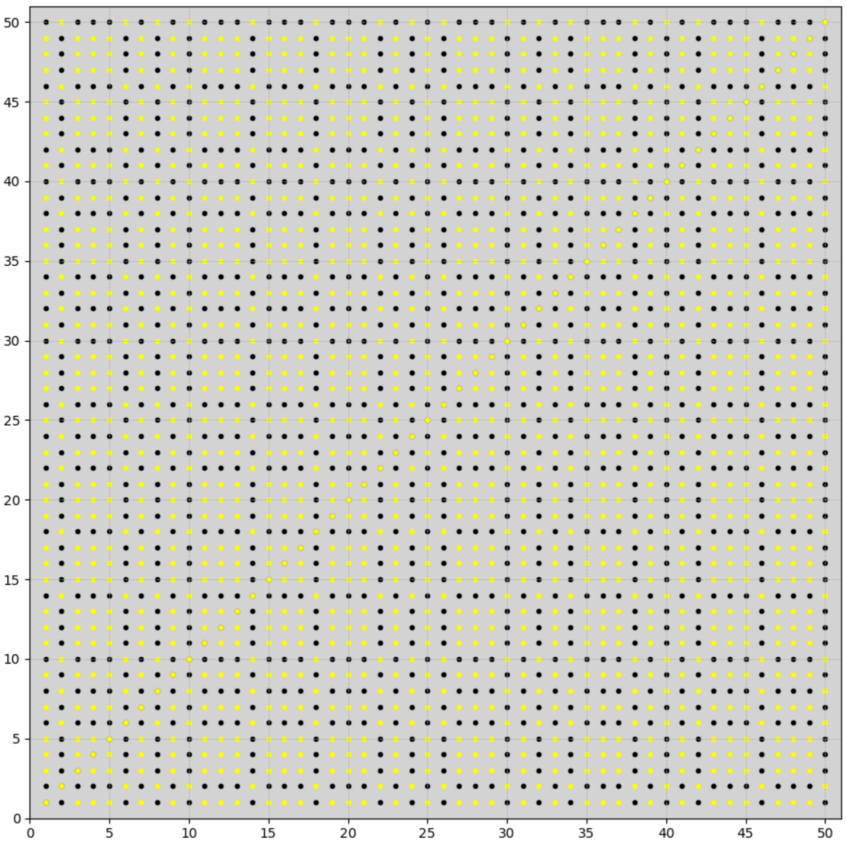
These two sequences appear in the context of game theory [2], where they are commonly referred to as **Vile** and **Dopey** numbers, respectively. These sequences are listed as A003159 and A036554 in the On-Line Encyclopedia of Integer Sequences (OEIS). They exhibit intriguing structural properties [1]: D, the complement of the set V, is formed by doubling each element of V. Thus, if a number n belongs to V, then $2n$ does not. The membership of the parameters x and y in these sequences plays a crucial role in determining the interior properties of $E(x, y)$.

Theorem 1. *The rule for determining whether $E(x, y)$ has empty or non-empty interior takes the following form:*

- *If x and y belong to the same set, either V or D, then $E(x, y)$ has empty interior.*
- *If one of the parameters belongs to V and the other to D, then $E(x, y)$ has non-empty interior.*

Our recent results yield a partial proof of the above theorem. In [3], it is shown using a probabilistic approach that if $\{0, x, y, x + y\}$ forms a complete residue system modulo 4, then $E(x, y)$ has positive Lebesgue measure. Since $E(x, y)$ is a self-similar set with similarity dimension 1, it follows from [4] that positive Lebesgue measure implies non-empty interior.

We visualize the discrete parameter space $\mathbb{N}^2$ as follows: a point (x, y) is colored yellow if $E(x, y)$ has empty interior, and black if it has non-empty interior.



By applying a series of transformations that preserve the topological structure of $E(x, y)$, we can extend our results to the case where $(x, y) \in \mathbb{Z}^2$ or $(x, y) \in \mathbb{Q}^2$.

Visualizing the solutions to the aforementioned problem yields a nontrivial tiling of the lattice $\mathbb{N}^2$. The tiling is aperiodic, i.e., there is no nontrivial translation in the allowed directions under which the tiling remains invariant. However, the pattern is not fully chaotic and displays a self-similar and scale-invariant structure. These properties are primarily governed by the specific features of Vile and Dopey numbers.

[1] Allouche J.-P., Arnold A., Berstel J., Brlek S., Jockusch W., Plouffe S., Sagan B. E., A relative of the Thue-Morse sequence, *Discrete Mathematics*, **139** (1995), no. 1-3, 455–461.

[2] Fraenkel A. S., The vile, dopey, evil and odious game players, *Discrete Mathematics*, **312** (2012), no. 1, 42–46.

[3] Makarchuk O., Karvatskyi D., On the Lebesgue measure of one generalised set of subsums of geometric series, *Mat. Stud.*, **62** (2024), no. 2, 115–120.

[4] Schief A., Separation properties for self-similar sets, *Proceedings of the American Mathematical Society*, **122** (1994), no. 1, 111–115.