On canonical factorization for a Poisson process on a finite Markov chain

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Let $\{x_t, t \geq 0\}$ be a regular finite Markov chain with the state space $\{1, \ldots, m\}$ and the intensity matrix $\mathbf{Q} = \|q_{ij}\|_{i,j=1}^m$. Given $x_t = i$, the process ξ_t evolves like a compound Poisson process with the rate of jumps $\lambda_i > 0$ and the density of jumps $f_i(x)$. Then $\{\xi_t, x_t\}$ is the Markov additive process with the cumulant function

$$\mathbf{K}[r] = \int_{-\infty}^{\infty} (e^{rx} - 1)\, \mathbf{\Lambda}\mathbf{f}(x)\, dx + \mathbf{Q}, \quad \Re(r) = 0,$$

where $\mathbf{\Lambda} = \|\lambda_i \delta_{ij}\|$ and $\mathbf{f}(x) = \|f_i(x)\, \delta_{ij}\|$ (see, for instance, [1]).

For $s > 0$, let $\mathbf{Z}(s) = s\mathbf{I} + \mathbf{\Lambda} - \mathbf{Q}$, where $\mathbf{I}$ is the identity matrix. Since the eigenvalues of the intensity matrix have non-positive real parts, the eigenvalues of $\mathbf{Z}(s)$ have strictly positive real parts. Consequently, $\mathbf{Z}(s)$ is nonsingular and we can define the nonsingular continuous matrix funciton

$$\mathbf{M}_s[r] = \mathbf{Z}^{-1}(s)\,(s\mathbf{I} - \mathbf{K}[r]) = \mathbf{I} - \mathbf{Z}^{-1}(s)\, \mathbf{\Lambda} \int_{-\infty}^{\infty} e^{rx}\mathbf{f}(x)\, dx, \quad \Re(r) = 0.$$

The elements of $\mathbf{M}_s[r]$ belong to the ring with identity of Fourier transforms of absolutely integrable functions. For sufficiently large s, matrix $\frac{1}{2}\left(\mathbf{M}_s[0] + (\mathbf{M}_s[0])^\top\right)$ has positive eigenvalues. Therefore, by combining Theorems 7.3 and 8.1 from [2], we obtain the statement.

Proposition 1. *There exists $s_0 > 0$ such that for any $s > s_0$ the matrix $\mathbf{M}_s[r]$ possesses a left canonical factorization*

$$\mathbf{M}_s[r] = \mathbf{M}_s^+[r]\, \mathbf{M}_s^-[r], \tag{1}$$

where $\mathbf{M}_s^+[r]$ and $\mathbf{M}_s^-[r]$ are matrix functions with elements from the subrings of Fourier transforms of functions with positive and negative supports, respectively. With the normalizing condition $\mathbf{M}_s^+[\infty] = \mathbf{I}$, the factors of the factorization are unique.

Remark 1. For the scalar case ($m = 1$), the analogue of the canonical factorization (1) on the axis $\Re(r) = 0$ was formulated in [3, Theorem 2.1] (see, Condition B_1). Moreover, [3, Theorem 3.3] established a link between the components of the canonical and infinitely divisible factorizations, and [3, Theorem 5.3] explored the extension of these components to a wider strip under Cramer's conditions on the jump distribution.

[1] Gusak D. Boundary problems for the processes with independent increments on the Markov chains and semi-Markov processes (in Ukrainian), Institute of Mathematics, Kyiv, 1998, 320 pp.

[2] Gokhberg I., Krein M., Systems of integral equations on a half line with kernels depending on the difference of arguments, *Am. Math. Soc., Transl., II. Ser.* **14** (1960), 217–287.

[3] Bratiychuk N., Gusak D., Boundary problems for processes with independent increments, Naukova Dumka, Kyiv, 1990, 264 pp.