Multistability and energy optimization in a gradient Kuramoto-type model

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We study a gradient oscillator model of Kuramoto type defined on a ring of N identical oscillators. The model is motivated by the configurational energy of a frustrated spin system, where spins interact ferromagnetically with nearest neighbors and antiferromagnetically with next-nearest neighbors. In angular coordinates $\theta_i \in \mathbb{S}^1$, the governing equations take the form

$$\dot{\theta}_i = a\sin(\theta_i - \theta_{i+1}) + a\sin(\theta_i - \theta_{i-1}) + b\sin(\theta_i - \theta_{i+2}) + b\sin(\theta_i - \theta_{i-2}), \quad i = 1, \dots, N, \quad (1)$$

where a and b are coupling strengths, all subscripts are assumed modulo N. Given the attractive and repulsive interactions between neighbouring and second-neighbouring oscillators, we choose $a < 0 < b$. The system (1) is a gradient flow with respect to the energy potential

$$E_a = \sum_{i=1}^{N} \Big(2 + \tfrac{a^2}{4} + 2a\cos(\theta_{i+1} - \theta_i) + 2\cos(\theta_{i+2} - \theta_i)\Big).$$

The circulant coupling structure of (1) ensures translational invariance of the system and leads to a rich family of equilibria in the phase-difference coordinates $\varphi_i = \theta_1 - \theta_{i+1}$, $i = 1, \dots, N-1$: full *synchronization* $\Phi^{\text{sync}} = (0, 0, \dots, 0)$, *rotating waves*

$$\Phi_k^{\text{rw}} = \Big(k\tfrac{2\pi}{N},\, 2k\tfrac{2\pi}{N},\, \dots,\, (N-1)k\tfrac{2\pi}{N}\Big), \quad k = 0, \dots, N-1,$$

anti-phase (π-states), as well as asymmetric isolated equilibria and manifolds of non-isolated equilibria. Note that $\Phi_0^{\text{rw}} = \Phi^{\text{sync}}$.

A central result of our analysis is the complete stability classification of all rotating waves. The stability type of each Φ_k^{rw} is fully determined by the wave number k, the size N, and the ratio a/b. In particular, certain ranges of k yield attractors or repellers independently of a/b, while the remaining ranges produce parameter-dependent transitions between attractor, saddle, and repeller regimes.

A notable consequence of this classification is that the system exhibits high multistability: for a fixed value of a/b, several rotating waves with different wave numbers k can coexist as attractors simultaneously. The number and identity of coexisting attractors change as a/b varies, giving rise to a rich bifurcation structure.

We also describe the energy landscape and the Maxwell set, i.e., the set of parameter values at which the global energy minimum transitions between different configurations. Without loss of generality, we fix $b = 1$ and treat $a < 0$ as the single free parameter.

Proposition 1. *As the negative parameter a increases to 0, the global minimum of E_a shifts sequentially from Φ_k^{rw} to Φ_{k+1}^{rw}, without skipping any rotating wave. The transition between Φ_k^{rw} and Φ_{k+1}^{rw} occurs at the critical value*

$$a_c(k) = -4\cos\Big(\tfrac{\pi(2k+1)}{N}\Big)\cos\Big(\tfrac{\pi}{N}\Big). \quad (2)$$

Moreover, the critical values (2) provide explicit bounds on the uniform phase difference $2\pi k/N$ that characterizes the global energy minimizer, thus directly linking the coupling parameter a to the energetically optimal configuration of the system.

In the thermodynamic limit $N \to \infty$, the discrete family of rotating waves becomes a continuous one-parameter family of equilibria $\Phi_\sigma^{\mathrm{rw}}$ parameterized by $\sigma \in [0,1)$, where $\sigma = k/N$. In this limit, the global energy minimizer is the synchronized state if $a < -4$, while for $-4 < a < 0$ the optimal uniform phase difference contracts to the single value

$$\Delta\theta^{\mathrm{opt}} = \arccos\left(\tfrac{-a}{4}\right).$$

This result is consistent with the known energy minimizers for frustrated spin chains established in $[1,2]$.

The stability transitions between coexisting attractors are accompanied by bifurcations of various types. Due to the high symmetry of the system, alongside standard local bifurcations one observes degenerate scenarios in which entire manifolds of equilibria emerge, including a *circular pitchfork bifurcation* giving rise to a one-dimensional invariant manifold of fixed points.

A further goal of this work is to provide a more detailed description of such bifurcations for low-dimensional cases and, where possible, to obtain a unified picture valid for general N. A natural subsequent direction is to extend the analysis from the ring geometry to a two-dimensional lattice, which is closer to the original physical setting of frustrated spin systems.

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