On the existence of regular K_3-regular graphs

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We study graphs that are simultaneously regular with respect to the ordinary vertex degree and regular with respect to the triangle degree (the number of K_3 subgraphs containing a given vertex). We call such graphs regular K_3-regular. Our research [2] focuses on the (non-)existence of these graphs with prescribed parameters (r_2, r_3), where r_2 is the vertex degree and r_3 is the triangle degree.

For any graph with parameters (r_2, r_3), the fundamental upper bound $r_3 \leq \binom{r_2}{2}$ must hold. By deriving technical relations between vertex and edge K_3-degrees, we established several non-existence results for broad range of these parameters. Specifically, we proved the following theorems.

Theorem 1. *There are no graphs with parameters* (r_2, r_3) *such that* $r_3 = \binom{r_2-1}{2} + c$ *for* $r_2 > 4$ *and* $0 < c < \frac{r_2-2}{2}$.

Theorem 2. *There are no graphs with parameters* (r_2, r_3) *such that* $r_3 = \binom{r_2}{2} - c$, *provided* $r_2 \geq 3$ *and* $1 \leq c \leq \frac{r_2-1}{2}$.

These results refine existing bounds in the literature [1, 3] and affirmatively answer open questions from [3] regarding the existence of vertex-girth-regular graphs of girth 3. And recently, Theorem 2 was extended to the case of girth 5 in [4].

Turán graphs, denoted as $\mathrm{Turan}(n, r)$, constitute a vital class of regular K_3-regular graphs. We established uniqueness results for specific parameter sets.

Theorem 3. *The graph* $\mathrm{Turan}(2m+2, m+1)$ *is the unique connected graph with parameters* $\left(2m, 4\binom{m}{2}\right)$, *where* $m \geq 2$.

Theorem 4. *Let* $r_2 \geq 7$. *If* $3 \mid r_2$, *then there exists a unique connected graph with parameters* $\left(r_2, \frac{r_2(r_2-3)}{2}\right)$, *namely* $\mathrm{Turan}(r_2+3, \frac{r_2}{3}+1)$. *If* $3 \nmid r_2$, *no graph with parameters* $\left(r_2, \frac{r_2(r_2-3)}{2}\right)$ *exists.*

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