Implicit linear difference equations with generalized periodic inhomogeneity over commutative rings

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We consider the implicit linear difference equation of m-th order with constant coefficients

$$a_m w_{n+m} + a_{m-1} w_{n+m-1} + \ldots + a_1 w_{n+1} + a_0 w_n = f_n,\ a_m \neq 0,\ n \in \mathbb{N}_0, \tag{1}$$

where the coefficients $a_0, a_1, \ldots, a_m$ and all elements of the sequence $\{f_n\}_{n=0}^{\infty}$ belong to the commutative ring R. This equation is implicit, since a_m could be non-invertible. We are looking for a sequence $\{w_n\}_{n=0}^{\infty} \in R^{\mathbb{N}_0}$ that satisfies this equation.

Definition 1. The sequence $\{f_n\}_{n=0}^{\infty}$ is called B-generalized periodic if for some $B = (b_1, b_2, \ldots, b_k)$ from the ring R it satisfies the recurrence relation $b_0 f_n + b_1 f_{n+1} + \ldots + b_k f_{n+k} = 0,\ n \in \mathbb{N}_0$.

Theorem 1. *Suppose that the sequence $\{f_n\}_{n=0}^{\infty}$ is B-generalized periodic for some $B = (b_0, b_1, \ldots, b_k)$. Let the homogeneous equation*

$$a_m w_{n+m} + a_{m-1} w_{n+m-1} + \ldots + a_1 w_{n+1} + a_0 w_n = 0,\ a_m \neq 0,\ n \in \mathbb{N}_0$$

has only a trivial solution in $R^{\mathbb{N}_0}$. Then if there exists the solution $\{w_n\}_{n=0}^{\infty}$ of the equation (1), it must also be B-generalized periodic for the same $B = (b_0, b_1, \ldots, b_k)$.

This theorem allows us to find the solutions in explicit form for a wide class of such difference equations, solving the linear system

$$\begin{cases} a_0 w_n + & a_1 w_{n+1} + & \ldots & + a_m w_{n+m} & = & f_n, \\ a_0 w_{n+1} + & a_1 w_{n+2} + & \ldots & + a_m w_{n+m+1} & = & f_{n+1}, \\ \vdots & & & & & \\ a_0 w_{n+k} + & a_1 w_{n+k+1} + & \ldots & + a_m w_{n+m+k} & = & f_{n+k}, \\ b_0 w_n + & b_1 w_{n+1} + & \ldots & + b_k w_{n+k} & = & 0, \\ \vdots & & & & & \\ b_0 w_{n+m-1} + & b_1 w_{n+m} + & \ldots & + b_k w_{n+m+k-1} & = & 0. \end{cases} \tag{2}$$

Let now R be the ring of p-adic integers $\mathbb{Z}_p$. Using this result and some previous results on equations of the type (1) (see [1]) one can also obtain sum of a convergent series $\sum\limits_{n=0}^{\infty} p^n f_n$ in the ring $\mathbb{Z}_p$, where f_n is B-generalized periodic.

Theorem 2. *Let us consider the $(b_0, ..., b_k)$-generalized periodic sequence of integers $\{f_k\}_{k=0}^{\infty}$. Then the series $\sum_{k=0}^{\infty} p^k f_k$ converges in p-adic topology to the element in p-localization of $\mathbb{Z}$*

$$\sum_{k=0}^{\infty} p^k f_k = \frac{f_0 B_{n-1} + f_1 p B_{n-2} + \ldots + f_{n-1} p^{n-1} B_0 + f_n p^n}{B_n} \in \mathbb{Z}_{(p)},$$

where $B_j = -1 + \sum_{k=0}^{j} b_{n-k} p^{k+1}$.

Let now $R = \mathbb{Z}$. Using the Pólya theorem about entire integer-valued function (see [2]) we can obtain the following theorem.

Theorem 3. *Let* $a_0 = 1$ *and for all roots* λ_i *of the characteristic polynomial of the equation* (1) *with* B*-generalized periodic inhomogeneity, the inequality*

$$|\ln \lambda_j| = \sqrt{\ln^2 |\lambda_j| + \arg^2 \lambda_j} < \ln 2$$

holds. Then there exists no more then one integer solution of the equation (1). *If* (1) *has an integer solution, then this solution satisfies the system* (2).

[1] Goncharuk A.B., Implicit linear difference equations over a non-Archimedean ring, *Visnyk of V.N.Karazin Kharkiv National University Ser. "Mathematics, Applied Mathematics and Mechanics"*, **93** (2021), p. 18–33.

[2] Cahen P.-J., Chabert J.-L., What You Should Know About Integer-Valued Polynomials, *The American Mathematical Monthly*, **123** (2016), no. 4, 311–337.