On the Absence of an Exceptional Set in the Main Relation of Wiman-Valiron Theory

S. I. Dubei[1], **A. I. Bandura**[2], **A. I. Mokhnal**[1], **O. B. Skaskiv**[1]
[1] Ivan Franko National University of Lviv, Lviv, Ukraine
[2]Ivano-Frankivsk National Technical University of Oil and Gas, Ivano-Frankivsk, Ukraine
sviatoslav.dubei@lnu.edu.ua, andriykopanytsia@gmail.com, smohnal9@gmail.com, oleh.skaskiv@lnu.edu.ua

The object of investigation is a class of functions analytic in the right half-plane. The half-plane contains all points whose real part is greater than some fixed a and for all x from the interval $(a, +\infty)$ the corresponding supremum of modulus of the function F inside some vertical strip is finite. There is selected subclass consisting of those functions, for which the right-hand derivative of the supremum logarithm tends to infinity. For these functions for which there exists an auxiliary function with some local behavior.

Let f be an entire function of the form

$$f(z) = f_0 + \sum_{k=1}^{+\infty} f_k z^k, \quad z \in \mathbb{C}. \tag{1}$$

For $r > 0$ we denote

$$M_f(r) = \max\{|f(z)|\colon |z| = r\}, \quad \mu_f(r) = \max\{|f_k|r^k\colon k \geq 0\}, \quad \nu_f(r) = \max\{k\colon |f_k|r^k = \mu_f(r)\}$$

the maximum modulus, maximal term and central index, respectively.

Let $S(a)$, $-\infty \leq a \leq +\infty$, be a class of functions analytic in $\Pi(a) = \{z : a < \operatorname{Re} z\}$ which are bounded in the vertical stripes, i.e.

$$(\forall x \in (a, +\infty))\colon\ M(x, F) := \sup\{|F(t + iy)| : a < t \leq x, y \in \mathbb{R}\} < +\infty.$$

By Maximum Modulus Principle the supremum in the vertical stripe is attained at its boundary, i.e. at the vertical line. Therefore, the boundedness in the vertical stripe is replaced by the boundedness on vertical line. Function

$$M(x, F) = \sup\{|F(x + iy)|\colon y \in \mathbb{R}\}$$

is non-decreasing on $(a, +\infty)$, and by Hadamard Three Lines Theorem the function $\ln M(x, F)$ is convex on $(a, +\infty)$ (see [2, p.14-16], [3, p.145, p.266]).

Therefore, for all $x \in (a, +\infty)$ there exists the non-decreasing right-hand derivative

$$L(x) = L(x, F) \overset{def}{=} (\ln M(x, F))'_+$$

at the interval $(a, +\infty)$.

Let us denote by $S_\infty(a)$ the class of the functions $F \in S(a)$ such that

$$L(x, F) \to +\infty \quad (x \to +\infty).$$

By S_0 we denote the class of functions $F \in S_\infty(0)$ for which there exists a function $\delta(x)\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties

1. $\delta(x) \nearrow +\infty \ \ (0 \le x \uparrow +\infty)$;
2. $\frac{L(x,F)}{\delta(x)} \nearrow +\infty \ \ (0 \le x \uparrow +\infty)$;
3. for all $x \ge x_0$ with sufficiently large x_0 one has

$$\left| L\left(x \pm \frac{\delta(x)}{L(x,F)}, F\right) - L(x,F) \right| \le \frac{L(x,F)}{\delta(x)}. \tag{2}$$

Note that class S_0 is non-empty. For every $p \in \mathbb{N}$ the function $F_p(z) = \exp\{\exp\{z^p\}\}$ belongs to the class S_0.

Theorem 1 ([1]). *Let $F \in S_0$. Then, for each $n \in \mathbb{N}$ the asymptotic relation*

$$F^{(n)}(w) = (1 + o(1))\, L^n(x,F)\, F(w) \qquad (x \to +\infty) \tag{3}$$

holds for every point w with $Re\, w = x$ such that the inequality

$$|F(w)| \ge \frac{M(x,F)}{1 + \varepsilon(x)},$$

is valid for a given positive function $\varepsilon(x)$ with $\varepsilon(x) \to +0$ $(x \to +\infty)$.

From Theorem 1 the following corollary is obtained.

Corollary 1 ([1]). *Let f be an entire transcendental function of the form (1), and $F(z) = f(e^z)$. If $F \in S_0$, then for given $n \in \mathbb{N}$ and for any point z, $|z| = r$, such that $|f(z)| = M_f(r)$, the asymptotic relation*

$$f^{(n)}(z) \sim \left(\frac{K_f(r)}{z}\right)^n \cdot f(z),$$

holds as $|z| = r \to +\infty$.

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