EXTREMAL PROBLEMS FOR POSITIVE (NEGATIVE) PARTS OF FUNCTIONS AND THE INEQUALITIES OF VARIOUS METRICS

D. E. Doroshenko[1], **V. O. Kofanov**[1]
[1]Oles Honchar Dnipro National University, Dnipro, Ukraine
daniil.e.doroshenko@gmail.com, vladimir.kofanov@gmail.com

Let L^r_∞ be the space of functions $x \in L_\infty(\mathbb{R})$ that have locally absolutely continuous derivatives up to order $(r-1)$, and such that $x^{(r)} \in L_\infty(\mathbb{R})$.

$f \in L^1_\infty$ is a comparison function for $x \in L^1_\infty$ if $\|x\|_\infty \le \|f\|_\infty$ and

$$x(\xi) = f(\eta) \Rightarrow |x'(\xi)| \le |f'(\eta)|.$$

An odd 2ω-periodic function $\varphi \in L^1_\infty$ is called an S-function if $\varphi(\cdot - \omega/2)$ is even, $|\varphi|$ is convex upward on $[0,\omega]$ and strictly increasing on $[0,\omega/2]$. Let

$$\|x_\pm\|_{p|\delta} := \sup\{\|x_\pm\|_{L_p(a,b)} : a,b \in \mathbb{R},\ \mu(\operatorname{supp}_{(a,b)} x_\pm) \le \delta\},$$

where $\operatorname{supp}_{(a,b)} x := \{t \in (a,b) : |x(t)| > 0\}$, $x_\pm := \max\{\pm x, 0\}$. Then

$$\|x\|_{p|\delta} := \max\{\|x_+\|_{p|\delta},\ \|x_-\|_{p|\delta}\}.$$

Denote by K_φ the class of functions $x \in L^1_\infty$, for which the S-function φ is a comparison function and set $K_\varphi(p|\varepsilon) := \left\{x \in K_\varphi : \|x\|_{p|\varepsilon} \le \|\varphi\|_{p|\varepsilon}\right\}$.

By the symbol W we denote the class of continuous, nonnegative and convex functions Φ, defined on $[0,\infty)$, such that $\Phi(0) = 0$.

For $M \subset L^1_\infty$ denote by $\tilde{M}$ the class of functions $x \in M$ such that for $p, \delta > 0$ each of the exact upper bounds

$$\sup\left\{\int_a^b \Phi\big(x^p_\pm(t)\big)\,dt :\ a,b \in \mathbb{R},\ \mu\big(\operatorname{supp}_{[a,b]} x_\pm\big) \le \delta\right\}, \Phi \in W,$$

is attained on some interval $[\alpha,\ \beta]$ (which depends on Φ and p, δ). *Set*

$$W^r_{p|\varepsilon}(A_0, A_r) := \{x \in L^r_\infty : \|x\|_{p|\varepsilon} \le A_0,\ \|x^{(r)}\|_\infty \le A_r\},$$

$$L^{r,\lambda}_{p|\varepsilon} := \{x \in L^r_\infty : \|x\|_{p|\varepsilon} = \|\varphi_{\lambda,r}\|_{p|\varepsilon}\,\|x^{(r)}\|_\infty\},$$

where φ_r is the Euler ideal spline of order r, and $\varphi_{\lambda,r}(t) := \lambda^{-r}\varphi_r(\lambda t)$. Let

$$I_\pm(\delta) := \left\{[a,b] \subset \mathbb{R} : \mu\big(\operatorname{supp}_{[a,b]} x_\pm\big) \le \delta\right\},\ x \in K_\varphi(p|\omega).$$

Theorem 1. *Let φ be a 2ω-periodic S-function, $p, \delta > 0$, and $\Phi \in W$. Then*

$$\sup_{[a,b] \in I_\pm(\delta)}\ \sup_{x \in \widetilde{K}_\varphi(p|\omega)} \int_a^b \Phi\big(x^p_\pm(t)\big)\,dt = \int_A^B \Phi\big(\varphi^p_\pm(t+\tau)\big)\,dt,$$

where the segment $[A,B]$ is such that $\mu\big(\operatorname{supp}_{[A,B]} \varphi_\pm\big) = \delta$, and τ is chosen from the condition

$$\varphi_\pm(A + \Theta + \tau) = \varphi_\pm(B - \Theta + \tau) = \|\varphi\|_\infty,$$

while Θ satisfies

$$\delta = n\omega + 2\Theta, \quad n \in \mathbb{N} \cup \{0\}, \quad 2\Theta \in [0,\omega).$$

Theorem 2. *Let φ be a 2ω-periodic S-function, $p > 0$, $\delta > 0$, and $q \ge p$. Then the Boyanov–Naidenov problem of finding the exact upper bound*

$$\sup_{x \in \widetilde{K}_\varphi(p|\omega)} \|x_\pm\|_{q|\delta}.$$

is equivalent to the problem of finding the exact constant C in the inequality of different metrics

$$\|x_\pm\|_{q|\delta} \le C\|x\|_{p|\omega}, \quad x \in \operatorname{con}\widetilde{K}_\varphi(p|\omega)$$

Here $\operatorname{con}\widetilde{K}_\varphi(p\,|\,\omega)$ *denotes the cone generated by the set* $\widetilde{K}_\varphi(p\,|\,\omega)$*, moreover* $C = \frac{\|\varphi\|_{q|\delta}}{\|\varphi\|_{p|\omega}}$.

The following results follow from Theorems 1 and 2.

Theorem 3. *Let* $r \in \mathbb{N}$*;* $A_0, A_r, p, \delta > 0$*;* $q \ge p$*,* $\omega = \pi/\lambda$*, where* λ *is determined by condition* $A_0 = A_r\|\varphi_{\lambda,r}\|_{p|\omega}$*. Then*

$$\sup_{x\in \widetilde{W}^r_{p|\omega}(A_0,A_r)} \|x_\pm\|_{q|\delta} = A_r\|\varphi_{\lambda,r}\|_{q|\delta}.$$

Theorem 4. *Under the assumptions of Theorem 3, the Boyanov–Naidenov problem of finding the exact upper bound*

$$\sup_{x\in \widetilde{W}^r_{p|\omega}(A_0,A_r)} \|x_\pm\|_{q|\delta}$$

is equivalent to the problem of finding the sharp constant $C = C(\lambda)$ *in the inequality*

$$\|x_\pm\|_{q|\delta} \le C\|x\|^\alpha_{p|\omega}\|x^{(r)}\|^{1-\alpha}_\infty, \quad x \in \widetilde{L}^{r,\lambda}_{p|\omega},$$

where, $\alpha = \frac{r+1/q}{r+1/p}$. Moreover, $C(\lambda) = \frac{\|\varphi_r\|_{q|\lambda\delta}}{\|\varphi_r\|^\alpha_{p|\pi}}$.

Denote by T_n the set of trigonometric polynomials of order not greater than n. Then let

$$T^\omega_n(A\,|\,p) := \{T \in T_n : \ \|T\|_{p|\omega} \le A\ \|\sin n(\cdot)\|_{p|\omega}\}, \ \ \omega = \pi/n.$$

Theorem 5. *Let* $n \in \mathbb{N}$*;* $A, p, \delta > 0$*;* $q \ge p$*,* $\omega = \pi/n$*. Then*

$$\sup\bigl\{\|T_\pm\|_{q|\delta} : T \in T^\omega_n(A\,|\,p)\bigr\} = A\|\sin n(\cdot)\|_{q|\delta}.$$

Theorem 6. *Under the assumptions of Theorem 5, the Boyanov–Naidenov problem of finding the exact upper bound*

$$\sup_{T\in T^\omega_n(A|p)} \|T_\pm\|_{q|\delta}$$

is equivalent to the problem of finding the sharp constant C *in the inequality of different metrics*

$$\|T_\pm\|_{q|\delta} \le C n^{1/p-1/q}\|T\|_{p|\omega}, \quad T \in T_n.$$

Moreover, $C = \frac{\|\sin(\cdot)\|_{q|n\delta}}{\|\sin(\cdot)\|_{p|\pi}}$.

Let $\sigma_{h,r}$ be the set of polynomial splines of order r of defect 1 with knots kh, $k \in \mathbb{Z}$, then let

$$\sigma^\varepsilon_{h,r}(A\,|\,p) := \{s \in \sigma_{h,r} : \ \|s\|_{p|\varepsilon} \le A\|\varphi_{\lambda,r}\|_{p|\varepsilon}\}, \ \ \lambda = \pi/h.$$

Theorem 7. *Let* $r \in \mathbb{N}$*;* $A, p, \delta, h > 0$*;* $q \ge p$*,* $\lambda = \pi/h$*. Then*

$$\sup\Bigl\{\|s_\pm\|_{q|\delta} : s \in \sigma^h_{h,r}(A\,|\,p)\Bigr\} = A\|\varphi_{\lambda,r}\|_{q|\delta}.$$

Theorem 8. *Under the assumptions of Theorem 6, the Boyanov–Naidenov problem of finding the exact upper bound*

$$\sup_{s\in \widetilde{\sigma}^h_{h,r}(A|p)} \|s_\pm\|_{q|\delta}$$

is equivalent to the problem of finding the sharp constant C *in the inequality of different metrics*

$$\|s_\pm\|_{q|\delta} \le \lambda^{1/p-1/q}\cdot C \cdot \|s\|_{p|h}, \quad s \in \widetilde{\sigma}_{h,r}.$$

Moreover, $C = \frac{\|\varphi_r\|_{q|\lambda\delta}}{\|\varphi_r\|_{p|\pi}}$.

[1] Doroshenko D.E., Kofanov V.O. The Boyanov–Naidenov problem and inequalities of various metrics. Ukr. Math. J., 2026, 78, No.3–4, 145–159.