Evolutionary-type inequalities for the products of the inner radii of multiply connected domains

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In the present work, we propose an approach for obtaining evolutionary-type estimates for the products of the inner radii of pairwise non-overlapping multiply connected domains with respect to fixed points of the complex plane for all values of a certain parameter $\gamma \in (0, n]$.

A function $g_B(z,a)$ which is continuous in $\overline{\mathbb{C}}$, harmonic in $B\backslash\{a\}$ apart from z, vanishes outside B, and in the neighborhood of a has the following asymptotic expansion

$$g_B(z,a) = -\ln|z-a| + \delta + o(1), \quad z \to a,$$

if $a = \infty$, then

$$g_B(z,\infty) = \ln|z| + \delta + o(1), \quad z \to \infty,$$

is called the (classical) Green function of the domain B with pole at $a \in B$. The Green function is an invariant under a conformal and univalent mapping f, i.e.,

$$g_{f(B)}(f(z), f(a)) = g_B(z,a), \quad z \in B \setminus \{a\}.$$

The inner radius $r(B,a)$ of the domain B with respect to a point a is the quantity $r(B,a) = e^{\delta}$.

Let

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$Y_n(\gamma) = r^{\gamma}\left(B_{\infty}, \infty\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where $n \in \mathbb{N}$, $\gamma \in (0, n]$, and $\{a_k\}_{k=1}^n$ is an arbitrary fixed system of distinct points in the complex plane $\mathbb{C}\backslash\{0\}$; B_0, B_{∞}, $\{B_k\}_{k=1}^n$, is an arbitrary system of pairwise non-overlapping domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_{\infty} \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, n]$, $\tau \in (0, \gamma)$. Then, for any fixed system of different points $\{a_k\}_{k=1}^n \in \mathbb{C}\backslash\{0\}$ and an arbitrary set of pairwise non-overlapping domains $\{B_k\}_{k=0}^n$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, the following inequality holds:*

$$I_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} I_n(\tau) \left(I_n(0)\right)^{-\frac{\gamma-\tau}{n}} \left(\prod_{k=1}^{n} |a_k|\right)^{\frac{2(\gamma-\tau)}{n}}.$$

Theorem 2. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, n]$, $\tau \in (0, \gamma)$. Then, for any fixed system of different points $\{a_k\}_{k=1}^n \in \mathbb{C}$ and an arbitrary set of pairwise non-overlapping domains B_{∞}, $\{B_k\}_{k=1}^n$, $\infty \in B_{\infty} \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequality holds:*

$$Y_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} Y_n(\tau) \left(Y_n(0)\right)^{-\frac{\gamma-\tau}{n}}.$$

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[2] Denega I., Zabolotnyi Y., Some extremal problems on the Riemannian sphere, *Carpathian Math. Publ.* **16** (2024), no. 2, 593 – 605.