Conditional symmetries and exact solutions of a reaction-diffusion system

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The study is devoted to the analysis of Lie and nonclassical (Q-conditional) symmetries for a system of two coupled cubic reaction-diffusion equations

$$\begin{aligned} u_t &= d_1 u_{xx} - A_1 u^2(1-u) - B_1 uv + 2B_1 u^2 v + B_1 uv^2, \\ v_t &= d_2 v_{xx} - B_2 v^2(1-v) - B_2 uv + 2B_2 uv^2 + A_2 u^2 v, \end{aligned} \tag{1}$$

where $d_i > 0$, while A_i and B_i, $i = 1, 2$ are nonnegative constants with $(A_i, B_i) \neq (0, 0)$.

Depending on the values of the coefficients d_i, A_i, and B_i, $i = 1, 2$, all Lie and Q-conditional symmetries of system (1) are identified [1]. A wide class of new exact solutions is constructed, including solutions expressible in terms of the Lambert W-function and solutions that cannot be obtained via Lie symmetries. In addition, a new application of the system to a real-world process is discussed.

The main results of our investigation are as follows.

Theorem 1. *The reaction-diffusion system (1) admits a Q-conditional symmetry of the form*

$$Q = \partial_t + \xi(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v$$

only in the case $d_1 \neq d_2$, $B_1 = B_2 = 0$ and $A_1 d_2 = A_2 d_1$, i.e.

$$\begin{aligned} u_t &= d_1(u_{xx} - Au^2(1-u)), \\ v_t &= d_2(v_{xx} + Au^2 v). \end{aligned} \tag{2}$$

The most general form of the Q-conditional symmetry of system (2) is given by the formula

$$Q = \partial_t + \gamma\partial_x + \Big(\alpha \exp(\sigma(x + d_2\sigma t))(u-1) + \beta v\Big)\partial_v,$$

where A, α, β, γ are arbitrary constants and $\sigma = \gamma\frac{d_2 - d_1}{2d_1 d_2}$.

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[1] Cherniha R., Broadbridge P., Davydovych V., Marquette I., Symmetries and exact solutions of a reaction-diffusion system arising in population dynamics, *Quaestiones Mathematicae*, in press, arXiv:2412.13097v2.