Skorokhod reflection problem on a graph

I. Chulakov[1], **A. Pilipenko**[2]

[1]Institute of Mathematics of NAS of Ukraine,
Dragomanov Ukrainian State University, Kyiv, Ukraine

[2]Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

igorchzpua@gmail.com, pilipenko.ay@gmail.com

In 1961, A. Skorokhod formulated the reflection problem in his seminal work [1], which subsequently proved to be an effective method for constructing reflected diffusions. This approach naturally defines the Skorokhod reflection mapping, which assigns a trajectory of the reflected process to a given unconstrained path. This mapping turned out to be highly useful for solving various problems in queueing theory (see, e.g., [2]). Over the years, the Skorokhod reflection problem has been generalized in various directions. In particular, A. Pilipenko [3] posed the problem of jump-like reflection, which allowed for the construction of a Brownian motion with jump-like reflection (see [4], [5]).

The aim of our work is to construct a similar mapping for natural gluing of functions on a coordinate graph, where the coordinate graph in $\mathbb{R}^m$ is $\{(x_1, \dots, x_m) \in \mathbb{R}^m : x_i x_j = 0 \text{ for } i \neq j\}$. To this end, we first introduce a graph switching problem. The function $\vec{X} : \mathbb{R}_+ \mapsto \mathbb{R}^m$ is called a solution to the graph switching problem for the pair of m-dimensional functions $\vec{W}$, $\vec{H}$ if its range lies on the coordinate graph and its coordinates can be represented in the following form:

$$X_i(t) = W_i\big(T_i(t)\big) + H_i\big(T_B(t)\big), \qquad 1 \le i \le m, \tag{1}$$

where T_i, T_B are time-change functions. They are non-decreasing and there are disjoint sets B and A_i, $1 \le i \le m$, such that $\mathbb{R}_+ = \bigcup_{i=1}^m A_i \cup B$ and

$$\int_0^\infty \mathbb{I}_{A_i}(t)\,\mathrm{d}T_B(t) = 0, \quad \int_0^\infty \mathbb{I}_{A_i}(t)\,\mathrm{d}T_j(t) = 0, \quad \int_0^\infty \mathbb{I}_B(t)\,\mathrm{d}T_i(t) = 0, \quad \text{for } i \neq j.$$

We explore the limit behavior of the solutions of the graph switching problem under the assumption that they spent no time on the negative half-axes in the limit. Furthermore, we assume that the measure of time intervals where the compensating time function $T_B^{(n)}$ increases converges to 0. We prove that the limit function is formed by the Skorokhod reflections of the functions W_i, properly glued together at the origin of the coordinate graph with shifted time scales. This result turns out to be useful for constructing the Walsh Brownian motion. We apply this result to the stochastic processes to get our main result which is formulated as follows:

Theorem 1. *Suppose that the sequence of the stochastic processes $\vec{X}^{(n)} \in D([0,\infty), \mathbb{R}^m)$ are solutions to a graph switching problem for the processes $\vec{W}^{(n)}$, $\vec{H}^{(n)}$ which satisfy the following conditions:*

1. *The sequence $\{(\vec{W}^{(n)}, \vec{H}^{(n)})\}_{n\ge1}$ converges weakly to some continuous process $(\vec{W}, \vec{H})$.*
2. *For all $T > 0$ and $1 \le i \le m$ we have*

$$\inf_{t\in[0,T]} \Big(X_i^{(n)}(t) \wedge 0\Big) \xrightarrow{P} 0, \qquad \sup_{t\in[0,T]\cap B^{(n)}} \Big(X_i^{(n)}(t) \vee 0\Big) \xrightarrow{P} 0.$$

3. *The sequence of stochastic processes* $H^{(n)}(\cdot) := \sum_{i=1}^m H_i^{(n)}(\cdot)$ *converges weakly to a continuous strictly increasing process* $H(\cdot)$ *and* $T^{(n)}(\cdot) := \sum_{i=1}^m T_i^{(n)}(\cdot)$ *converges weakly to the identity function* $\mathrm{Id}(\cdot)$ *on* $\mathbb{R}_+$ *as* $n \to \infty$.

4. *Let* $L_i := -\inf_{s\in[0,t]}(W_i(s) \wedge 0)$. *denote the running minimum of the process* W_i. *Then the pair of functions* $L_i(\cdot)$ *and* $H_i(\cdot)$ *do not have common levels of constancy.*

5. *For all* $i \neq j$ *from* $\{1, \dots, m\}$ *the pair of functions* $H_i^{-1}(L_i(\cdot))$ *and* $H_j^{-1}(L_j(\cdot))$ *do not have common levels of constancy. Here* H_i^{-1} *denotes the generalized inverse function defined as follows:* $f^{-1}(t) := \inf\{s \geq 0 \mid f(s) > t\}$, $t \in \mathbb{R}$.

Then the sequence of processes $\vec{X}^{(n)}$ *converges weakly to the process* $\vec{X}$ *represented by the formula*

$$X_i(t) = W_i(T_i(t)) + H_i(T_B(t)) = W_i(T_i(t)) + L_i(T_i(t)) =: W_i^{\text{reflected}}(T_i(t)), \tag{2}$$

where T_i, T_B *are the unique solution of the system*

$$\begin{cases} \displaystyle\sum_{i=1}^m T_i(t) = t, \\ L_i(T_i(t)) = H_i(T_B(t)), \quad 1 \leq i \leq m. \end{cases} \tag{3}$$

These results motivate us to give the following definition of the Skorokhod reflection problem on a graph:

Definition 1. Let $\vec{W} = (W_1, \dots, W_m) \in D([0,\infty), \mathbb{R}^m)$, and $\vec{H} = (H_1, \dots, H_m) \in D([0,\infty), \mathbb{R}_+^m)$, where H_i are some non-decreasing functions. Let $\vec{H}(0) = \vec{0}$ and $\vec{W}(0)$ belongs to the positive part of the coordinate graph. We say that the function $\vec{X}$ is a solution to the *Skorokhod reflection problem on a graph* for a function $\vec{W}$ and a compensating function $\vec{H}$ if the coordinates of the function $\vec{X} = (X_1, \dots, X_m)$ can be represented by the formula (2), where functions T_i, T_B are solutions to the system (3).

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