On interpolation of some spaces associated with abstract and elliptic operators

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Let X and Y be quasi-Banach (resp., Banach) spaces, Υ be a linear topological space and $T : X \to \Upsilon$ be a continuous linear operator. All linear spaces are supposed to be complex. We assume that $Y \hookrightarrow \Upsilon$, i.e. Y is continuously embedded in Υ. We consider the linear space

$$X(T, Y) := (X)(T, Y) := \{u \in X : Tu \in Y\} \tag{1}$$

endowed with the graph quasi-norm

$$\|u\|_{X(T,Y)} := \|u\|_X + \|Tu\|_Y. \tag{2}$$

This quasi-normed space is complete and does not depend on Υ. If X and Y are Banach spaces, then so is $X(T, Y)$.

Let $\mathfrak{F}[\cdot, \cdot]$ be an arbitrary interpolation functor defined on the category of all interpolation pairs of quasi-Banach spaces (resp., Banach spaces).

We formulate sufficient conditions under which the result of the interpolation between spaces of type (1), (2) is the space of the same type.

Theorem 1. *Suppose that six quasi-Banach spaces (resp., Banach spaces) X_0, Y_0, Z_0, X_1, Y_1, and Z_1 and three linear mappings T, R, and S are given and satisfy the following conditions:*

(i) The pairs $[X_0, X_1]$ and $[Y_0, Y_1]$ are interpolation ones.

(ii) The spaces Y_0, Z_0, Y_1, and Z_1 are linear manifolds in a certain linear space.

(iii) The continuous embeddings $Y_0 \hookrightarrow Z_0$ and $Y_1 \hookrightarrow Z_1$ hold true.

(iv) The mapping T acts from $X_0 + X_1$ to $Z_0 + Z_1$, and its restriction to X_j is a bounded operator $T : X_j \to Z_j$ whenever $j \in \{0, 1\}$.

(v) The mapping R acts from $Z_0 + Z_1$ to $X_0 + X_1$, and its restriction to Z_j is a bounded operator $R : Z_j \to X_j$ whenever $j \in \{0, 1\}$.

(vi) The mapping S acts on Z_0+Z_1, and its restriction to Z_j is a bounded operator $S : Z_j \to Y_j$ whenever $j \in \{0, 1\}$.

(vii) The equality $TRw = w + Sw$ is valid for every $w \in Z_0 + Z_1$.

Then the equality of spaces

$$\mathfrak{F}[X_0(T, Y_0), X_1(T, Y_1)] = (\mathfrak{F}[X_0, X_1])(T, \mathfrak{F}[Y_0, Y_1]) \tag{3}$$

holds true up to equivalence of quasi-norms, with these spaces being well defined.

This result extends the theorems by J.-L. Lions and E. Maganes (1968), V.A. Mikhailets and A.A. Murach (2006), N. Kalton, S. Mayboroda and M. Mitrea (2007) to general interpolation functors.

Consider an application of Theorem 1 to some distribution spaces $X(A,Y)$, where X and Y are continuously embedded in $\Upsilon := \mathcal{D}'(\Omega)$, Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, with $n \geq 2$, and $T := A$ is an elliptic differential operator on $\overline{\Omega}$ with complex-valued coefficients of class $C^\infty(\overline{\Omega})$. (As usual, $\mathcal{D}'(\Omega)$ is the linear topological space of all distributions on Ω.)

Suppose that X is either the Besov space $B^s_{p,q}(\Omega)$ or Triebel–Lizorkin space $F^s_{p,q}(\Omega)$ with $s \in \mathbb{R}$, $0 < p \leq \infty$ ($p \neq \infty$ for F-spaces), and $0 < q \leq \infty$. They are Banach iff $p \geq 1$ and $q \geq 1$. Let $\mathcal{U}(\Omega)$ be the class of all these spaces. If $\dim \Omega = 2$, we suppose in addition that A is properly elliptic. Then the order of A is an even number $2\ell \geq 2$.

A quasi-Banach space $Y \hookrightarrow \mathcal{D}'(\Omega)$ is said to be admissible if $W \hookrightarrow Y$ for a certain space W from the class $\mathcal{U}(\Omega)$.

Theorem 2. *Let X_0 and X_1 be quasi-Banach (resp., Banach) spaces from the class $\mathcal{U}(\Omega)$, and let Y_0 and Y_1 be admissible quasi-Banach (resp., Banach) spaces embedded continuously in $\mathcal{D}'(\Omega)$. Given $j \in \{0,1\}$, we put $Z_j := B^{s-2\ell}_{p,q}(\Omega)$ if $X_j := B^s_{p,q}(\Omega)$ and put $Z_j := F^{s-2\ell}_{p,q}(\Omega)$ if $X_j := F^s_{p,q}(\Omega)$ (s, p, and q depends on j). Then*

$$\mathfrak{F}[X_0(A,Y_0), X_1(A,Y_1)] = (\mathfrak{F}[X_0,X_1])(A, \mathfrak{F}[Y_0 \cap Z_0, Y_1 \cap Z_1])$$

with equivalence of quasi-norms.

If $\mathfrak{F}[\cdot,\cdot] = (\cdot,\cdot)_{\theta,q}$ is the real interpolation functor and if X_0, X_1, Y_0, Y_1 are Besov spaces, then Theorem 2 leads to the following result:

Theorem 3. *Let $s_0, s_1, \alpha_0, \alpha_1 \in \mathbb{R}$, $s_0 \neq s_1$, $\alpha_0 \neq \alpha_1$, and $p, \beta, q_0, q_1, \gamma_0, \gamma_1 \in (0;\infty]$. Suppose also that*

$$B^{\alpha_0}_{\beta,\gamma_0}(\Omega) \hookrightarrow B^{s_0-2l}_{p,q_0}(\Omega) \quad \textit{and} \quad B^{\alpha_1}_{\beta,\gamma_1}(\Omega) \hookrightarrow B^{s_1-2l}_{p,q_1}(\Omega).$$

Assume that the interpolation parameters satisfy $0 < \theta < 1$ and $0 < q \leq \infty$, and put $s := (1-\theta)s_0 + \theta s_1$ and $\alpha := (1-\theta)\alpha_0 + \theta\alpha_1$. Then

$$\left(B^{s_0}_{p,q_0}(\Omega)(A, B^{\alpha_0}_{\beta,\gamma_0}(\Omega)), B^{s_1}_{p,q_1}(\Omega)(A, B^{\alpha_1}_{\beta,\gamma_1}(\Omega))\right)_{\theta,q} = B^s_{p,q}(\Omega)(A, B^\alpha_{\beta,q}(\Omega))$$

up to equivalence of quasi-norms. This formula remains true if $\alpha_0 = \alpha_1$ and $\gamma_0 = \gamma_1$.

This Theorem allows proving the Fredhoml property of elliptic boundary-value problems on appropriate pairs of Besov spaces of low regularity. Theorem 2 applied to $\pm$-method of interpolation allows proving this property for Tribel-Lizorkin spaces of low regularity.

These results are obtained together with O.O. (A.A.) Murach [1].

[1] Chepurukhina I., Murach A., Distribution spaces associated with elliptic operators, 2024, 42 pp., arXiv:2406.08150