Rapidly varying solutions of second order differential equations with the product of regularly and rapidly varying nonlinearities of an unknown function and its first-order derivative

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We consider the second-order differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y')\varphi_1(y), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$ determines the sign of the right-hand side, the function $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ is continuous, and the functions $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous on the intervals Δ_{Y_i}, which represent a one-sided neighborhood of the points $Y_i \in \{0, \pm\infty\}$.

Furthermore, we assume that the function $\varphi_1 \colon \Delta_{Y_1} \to]0, +\infty[$ is regularly varying (see [1], p. 17) of order σ_1 as $y \to Y_1$, and the function $\varphi_0 \colon \Delta_{Y_0} \to]0, +\infty[$ is twice continuously differentiable on Δ_{Y_0} and satisfies the following conditions:

$$\lim_{s\to Y_0} \varphi_0(s) \in \{0, +\infty\}, \quad \varphi_0'(s) \neq 0, \quad \lim_{s\to Y_0} \frac{\varphi_0(s)\varphi_0''(s)}{(\varphi_0'(s))^2} = 1. \tag{2}$$

Conditions (2) imply that the function φ_0 and its first derivative are rapidly varying as the argument tends to Y_0 (see [2], pp. 91–92).

Definition 1 ([3]). A solution y of equation (1), defined on $[t_0, \omega[\subset [a, \omega[$, is called a $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the following conditions hold:

$$y^{(i)} : [t_0, \omega[\longrightarrow \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \quad (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0. \tag{3}$$

The theoretical framework for investigating this class of solutions is based on the general classification of solutions for n-th order Emden-Fowler type differential equations [3]. Within this classification, $P_\omega(Y_0, Y_1, \lambda_0)$-solutions are divided into four disjoint classes depending on the value of the parameter λ_0. For each case, specific a priori asymptotic properties have been established [3], allowing for the identification of the solution dynamics as $t \uparrow \omega$.

This work focuses on the existence conditions for a special class of $P_\omega(Y_0, Y_1, 1)$-solutions. The specificity of the $\lambda_0 = 1$ case is that these solutions and their first derivatives are rapidly varying functions as $t \uparrow \omega$. This property leads to a fundamental difference in research methodology compared to the non-singular cases and the case $\lambda_0 = \pm\infty$ discussed in [4,5]. Unlike those cases, $\lambda_0 = 1$ requires the application of the theory of rapidly varying functions and specialized asymptotic integration techniques.

We establish the existence conditions for rapidly varying $P_\omega(Y_0, Y_1, 1)$-solutions of equation (1) containing a product of regularly and rapidly varying nonlinearities. We provide explicit asymptotic representations for the discovered solutions and their first derivatives in a neighborhood of the limit point ω. Furthermore, we determine the number of such solutions based on the requirements for the equation coefficients. These results, expanding upon the approaches recently explored in [6], provide a novel basis for further research into nonlinear second-order equations.

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