On the stability of the maximum term of functional series in a system of functions

A. Yu. Bodnarchuk[1], **M. M. Dolynyuk**[2], **O. B. Skaskiv**[3]
[1,3]Ivan Franko National University of Lviv, Lviv, Ukraine
[2]Markiyan Shashkevych Brody profess. pedagog. college, Brody, Lviv region, Ukraine
8andriy1111@gmail.com, m.dolyniuk@brodypk.ukr.education, olskask@gmail.com

Let us denote by L_+ the class of positive continuous on $\mathbb{R}_+ := [0, +\infty)$ functions $l(t)$ such that $l(t) \uparrow +\infty$ $(t \to +\infty)$, and by $\mathcal{W}$ we denote the class of functions $w \in L_+$ such that $\int_1^{+\infty} x^{-2} w(x)dx < +\infty$.

Let $\mathcal{S}(f, \Lambda)$ be the class of positive convergent for all $x \geq 0$ the functional series of the form

$$F(x) = \sum_{k=0}^{+\infty} a_k f(x\lambda_k), \qquad (1)$$

where $\Lambda = (\lambda_k)$ is a sequence of non-negative numbers such that $\lambda_k \neq \lambda_j$ for all $k \neq j$, $a_k \geq 0$ $(k \geq 0)$, and a positive increasing to $+\infty$ function f on $[0; +\infty)$ such that $f(0) = 1$ and $\ln f(x)$ is a convex function on the same interval. By $\mathcal{S}_*(f, \Lambda)$ we denote the class of formal series of form (1) such that $a_n f(x\lambda_n) \to 0$ $(n \to +\infty)$ for every $x \in \mathbb{R}_+$, i.e., for every $x \in \mathbb{R}_+$ there exists the maximal term $\mu(x, F) = \max\{|a_n| f(x\lambda_n) \colon n \geq 0\} < +\infty$. In the case $f(x) \equiv e^x$, we denote $\mathcal{D}_*(\Lambda) := \mathcal{S}_*(f, \Lambda)$.

For a function $w \in L$ let us denote $B_w(x) = \sum_{n=0}^{+\infty} a_n e^{w(\lambda_n)} f(x\lambda_n)$.

From Theorem 2 and Theorem 3 ([3]), proved for entire multiple Dirichlet series, it follows the following statement.

Theorem A ([3], Theorem 2). *Let $w \in L$, $B_w \in D_*(\lambda)$ and condition*

$$\int_0^{+\infty} t^{-2} \ln \nu_0(t) dt < +\infty \qquad (2)$$

satisfies, where $\nu_0(t) = \int\limits_0^t e^{w(x)} dn(x)$, $n(x) = \sum_{\lambda_n \leq x} 1$. Then relation

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B_w) \qquad (3)$$

holds as $x \to +\infty$ outside some set $E \subset [0; +\infty)$, $\operatorname{meas} E < +\infty$.

Theorem A implies the following corollary.

Corollary 1. *Let $F \in \mathcal{D}_*(\Lambda)$. If there exists a function $w \in L$ such that $F_w \in \mathcal{D}_*(\Lambda)$, $\ln \nu \in \mathcal{W}$ (here $\nu(t) = \sum_{\lambda_n \leq t} e^{w(\lambda_n)}$) and*

$$e^{-w(\lambda_n)} \leq b_n \leq e^{w(\lambda_n)} \quad (n \geq k_1), \qquad (4)$$

then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesque measure such that

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B_+) = (1 + o(1)) \ln \mu(x, B_-) \qquad (5)$$

as $x \to +\infty$ $(x \notin E)$.

Other versions of these statements were previously proven in the paper [1].

Let us denote

$$\nu_0(t) = \nu\{u \geq 0\colon \ln f(u) \leq t\}, \quad \nu(G) = \sum\nolimits_{\lambda_n \in G} e^{w(\lambda_n)}$$

for every bounded set $G \subset \mathbb{R}_+$, where $w \in L_+$. The main result of the article [1] is the following statement: If there exists a function $w \in L_+$ such that $a_n e^{w(\lambda_n)} f(\lambda_n x) \to 0$ for every $x > 0$, $\ln \nu_0 \in \mathcal{W}$, then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesque measure such that the asymptotic relation $\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, F_w)$ holds as $x \to +\infty$ outside the set E.

For a series $F \in \mathcal{S}(f, \Lambda)$ and any sequence (b_n), $b_n \in \mathbb{R}_+ \setminus \{0\}$ $(n \geq 0)$ we consider

$$B^+(x) = \sum_{n=0}^{+\infty} a_n b_n f(x\lambda_n), \quad B^-(x) = \sum_{n=0}^{+\infty} a_n b_n^{-1} f(x\lambda_n).$$

We call that a series of the form (1) (maximal term of the series) **is stable** if the relations

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B^+) = (1 + o(1)) \ln \mu(x, B^-)$$

hold as $x \to +\infty$ outside some set $E \subset [0, +\infty)$ of the finite Lebesgue measure, i.e. $\operatorname{meas} E := \int_E dx < +\infty$.

The following theorem is true.

Theorem 1. *Let $F \in \mathcal{S}_*(f, \Lambda)$. If there exists a function $w \in L_+$ such that $B_w \in \mathcal{S}_*(f, \Lambda)$, $\ln \nu_0 \in \mathcal{W}$ and the inequalities (4) hold, then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesgue measure such that the relation (5) holds as $x \to +\infty$ $(x \notin E)$.*

To prove Theorem 1 we use the following statement from [2, Corollary 1]. We consider the class $\mathcal{I}(\nu, f)$ of functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ represented by integrals of the form

$$F(x) = \int_0^{+\infty} g(t) f(tx) \nu(dt),$$

where ν is a locally finite measure on $\mathbb{R}_+$, g is a positive ν-dimensional function, f is a positive function such as above.

Lemma 1 ([2]). *If condition (2) holds with $\nu_0(t) = \nu\big(\{u \geq 0\colon \ln f(u) \leq t\}\big)$, then for every function $F \in \mathcal{I}(\nu, f)$ there exists a set E of finite Lebesgue measure such that the asymptotic relation $\ln F(x) \leq (1 + o(1)) \ln \mu(x, F)$ holds as $x \to +\infty$ $(x \notin E)$, where $\mu(x, F) = \sup\{g(t) f(tx)\colon x \in \operatorname{supp} \nu\}$ and $\operatorname{supp} \nu$ is the support of the measure ν.*

The research was funded by the National Research Foundation of Ukraine (project 2025.07/0427, "Newest complex probabilistic methods for studying asymptotic properties of analytic solutions of differential equations represented by multiple random series and integrals and their potential applications", 0126U002547).

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