ON LOWER AND UPPER ESTIMATES OF ZEROS FOR THE SECOND DERIVATIVE OF THE LINDELÖF FUNCTION

A. I. Bandura, D. I. Tsap, V. M. Hrytsuliak

Department of Physics and Mathematics, Ivano-Frankivsk National Technical University of Oil and Gas, Ivano-Frankivsk, Ukraine

andriy.bandura@nung.edu.ua, denys.tsap-ip232@nung.edu.ua,
vitalii.hrytsuliak-ip253@nung.edu.ua

The paper considers lower and upper estimates of the second derivative of a class of entire functions known as the Lindelöf functions. These functions admit the representation in the form of the infinite product

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^a}\right), \qquad a > 1,\ z \in \mathbb{C}. \tag{1}$$

These functions are related with trigonometric functions. In particular, for $a = 2$ we have

$$f(z^2) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi z}.$$

In 1968, B. Lepson [1] introduced the notion of bounded index to study local and asymptotic behaviour of entire functions of a complex variable. Later, A.C. King and S.M. Shah [2] examined index boundedness for the Lindelöf functions. In their work it was necessary to approximate lower and upper bounds for zeros of the first derivative of these functions. A.C. King and J.D. Patterson [3] later proposed a computer method for locating such bounds with higher numerical accuracy.

Theorem 1 ([3]). *Let* $H_1(x) = \frac{f'(x)}{f(x)}$*, where* f *is defined in* (1)*,* $x \in \mathbb{R}$ *and*

$$H_{1l}(x) = \sum_{n=1}^{k} \frac{1}{x - n^a} - \frac{a + k - 1}{(a-1)(k+1-x)^a},$$

$$H_{1u}(x) = \sum_{n=1}^{k} \frac{1}{x - n^a} - \frac{1}{(a-1)(k+1)^{a-1}}.$$

If $k \in \mathbb{N}$ *is chosen such that* $x < k + 1$*,* $a > 1$*, then for each* n *with* $n^a < x < (n+1)^a$*, we have*

$$H_{1l} < H < H_{1u}$$

The functions H_{1u}*,* H_{1l} *are decreasing functions of* x *in each interval.*

Their estimates are based on Laguerre's Theorem: *if* $f(z)$ *is an entire function which is real for real* z *and has only real zeros, then the zeros of* $f'(z)$ *are also real and are separated by the zeros of* $f(z)$*.*

For higher derivatives the authors mentioned that the arguments used for the first derivative can also be applied, but the detailed proof and estimates were omitted. The aim of this work is **to obtain estimates for the zeros of the second derivative of the Lindelöf functions.**

Using results obtained for the first derivative of the Lindelöf functions and applying Laguerre's theorem, we obtain bounds for the second derivative. Our main result is the following:

Theorem 2. *Let* $H_2(x) = \frac{f''(x)}{f'(x)}$*, where* f *is the Lindelöf function. Let* x_n *be the* n*-th zero of* $f'(x)$*. Then for* $x \in (x_n, x_{n+1})$ *the inequality*

$$H_{2l}(x) < H_2(x) < H_{2u}(x)$$

holds, where

$$H_{2l}(x) = \sum_{j=1}^{k} \frac{1}{x - x_j} - \frac{a+k-1}{(a-1)(k+1-x)^a},$$

$$H_{2u}(x) = \sum_{j=1}^{k} \frac{1}{x - x_j} - \frac{1}{(a-1)(k+1)^{a-1}}.$$

The functions H_{2l} *and* H_{2u} *are decreasing on each interval.*

Based on these theoretical estimates we propose a computer method for calculating the zeros of the second derivative of the Lindelöf functions.

For solving nonlinear equations we used two-step Argyros–Shakhno–Yarmola iterative method [4]

$$z_k = x_k - \frac{2x_k - 2x_{k-1}}{f(2x_k - x_{k-1}) - f(x_{k-1})} f(x_k),$$

$$x_{k+1} = z_k - \frac{2z_k - 2x_k}{f(2z_k - x_k) - f(x_k)} f(z_k).$$

Using this iterative scheme together with theoretical bounds we obtain an algorithm for computing approximations of the n-th zero of the second derivative of the Lindelöf function.

The research of A. Bandura and V. Hrytsuliak was funded by the National Research Foundation of Ukraine (project 2025.07/0427, "Newest complex probabilistic methods for studying asymptotic properties of analytical solutions of differential equations represented by multiple random series and integrals and their potential applications", 0126U002547).

[1] Lepson B., Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index, *Proc. Sympos. Pure Math.* **11** (1968), 298–307.

[2] King A.C., Shah S.M., Indices of Lindelöf functions and their derivatives, *Rocky Mountain J. Math.* **2** (1972), 4, 579–593.

[3] King A.C., Patterson J.D. A computer method for approximating the zeros of certain entire functions, *Rocky Mountain J. Math.* **17** (1987), 1, 135–140.

[4] Argyros I.K., Shakhno S.M., Yarmola H.P. On the convergence of Kurchatov-type methods using recurrent functions for solving equations, *Mat. Stud.* **58** (2022), 1, 103–112.