
FIXED POINTS THEOREMS IN HAUSDORFF M -DISTANCE SPACES

V.F. Babenko¹, V.V. Babenko², O.V. Kovalenko¹

¹Oles Honchar Dnipro National University, Dnipro, Ukraine

²Drake University, Des Moines, USA

babenko.vladislav@gmail.com, vira.babenko@drake.edu, olegkovalenko90@gmail.com

We prove fixed point theorems in a space with a distance function that takes values in a partially ordered monoid. On the one hand, such an approach allows one to generalize some fixed point theorems in a broad class of spaces, including metric and uniform spaces. On the other hand, compared to the so-called cone metric spaces and K -metric spaces, we do not require that the distance function range has a linear structure. We also consider several applications of the obtained fixed point theorems. In particular, we consider the questions of the existence of solutions of the Fredholm integral equation in semi-linear metric spaces.

Definition 1. A set M with associative binary operation $+$ is called a monoid, if there exists $\theta_M \in M$ such that $\theta_M + x = x = x + \theta_M$ for all $x \in M$.

Definition 2. A monoid M is called a partially ordered monoid, if it is a partially ordered set, $M_+ := \{x \in M : \theta_M \leq x\} \neq \{\theta_M\}$, and the following condition holds:

$$\text{if } x_1 \leq y_1 \text{ and } x_2 \leq y_2, \text{ then } x_1 + x_2 \leq y_1 + y_2.$$

Definition 3. For a partially ordered monoid M denote by $Z(M)$ a family of sequences $\{x_1, \dots, x_n, \dots\} \subset M_+$ such that the following properties hold:

1. if $\{\theta_M, \dots, \theta_M, \dots\} \in Z(M)$ and $x \in M_+$, $\{x, \dots, x, \dots\} \in Z(M)$, then $x = \theta_M$;
2. if $\{x_n\}, \{y_n\} \in Z(M)$, then $\{x_n + y_n\} \in Z(M)$;
3. if $\{x_n\} \in Z(M)$ and $\theta_M \leq y_n \leq x_n$ for all $n \in \mathbb{N}$, then $\{y_n\} \in Z(M)$;
4. each subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\} \in Z(M)$ belongs to $Z(M)$.

$Z(M)$ will be called a family of null sequences in M .

Definition 4. We say that a series $\sum_{k=1}^{\infty} x_k$ is a Cauchy series, if for all increasing sequences of natural numbers $\{n_k\}$ and $\{m_k\}$ such that $m_k \geq n_k$ for all $k \in \mathbb{N}$, one has $\{\sum_{s=n_k}^{m_k} x_s\} \in Z(M)$. By \mathcal{CS} we denote the set of all sequences $\{x_n\} \subset M$ such that $\sum_{k=1}^{\infty} x_k$ is a Cauchy series.

The notion of a Cauchy series is enough for our purposes. It is related to but different from the notion of a convergent series, which is well studied for normed linear spaces. Notice that the usual definition for convergence of a series requires a definition for convergence of an arbitrary sequence of elements. The introduced notion of a Cauchy series uses a family of null sequences in M , which in fact determines only convergence of sequences to θ_M . Of course in the presence of a linear structure in M , convergence to an arbitrary element can be reduced to the convergence to θ_M . In the case of a rather arbitrary set M , this is not the case.

Definition 5. Let X be a set and M be a partially ordered monoid. A mapping $d_{X,M}(\cdot, \cdot): X \times X \rightarrow M_+$ such that $d_{X,M}(x, y) = d_{X,M}(y, x)$ for all $x, y \in X$ is called an M -valued dislocated distance in X , if $d_{X,M}(x, y) = \theta_M \implies x = y$. The pair $(X, d_{X,M})$ is called a dislocated M -distance space (DM -distance spaces for brevity).

Definition 6. Let $(X, d_{X,M})$ be a DM -distance space, and let a family of null sequences $Z(M)$ in M_+ be chosen. The triple $(X, d_{X,M}, Z(M))$ is called an equipped DM -distance space (EDM -distance space for short).

Definition 7. Let $(X, d_{X,M}, Z(M))$ be an EDM -distance space. We say that a sequence $\{x_n\} \subset X$ converges to $x \in X$, and write $x_n \rightarrow x$ as $n \rightarrow \infty$, if $\{d_{X,M}(x_n, x)\} \in Z(M)$.

Definition 8. We denote by \mathcal{CW} the family of all sequences $\{x_n\} \subset X$ such that

$$\{d_{X,M}(x_n, x_{n+1})\} \in \mathcal{CS}.$$

Definition 9. We call an EDM -distance space $(X, d_{X,M}, Z(M))$ \mathcal{W} -complete, if each sequence $\{x_n\} \in \mathcal{CW}$ converges to an element from the space X .

We prove the following theorem [1].

Theorem 1. Let $(X, d_{X,M}, Z(M))$ be a Hausdorff equipped M -distance space, $f: X \rightarrow X$ be weakly orbitally continuous, and $\{\lambda_n\}$ be a sequence of non-decreasing operators $\lambda_i: M_+ \rightarrow M_+$. Then f has a fixed point if $(X, d_{X,M}, Z(M))$ is \mathcal{W} -complete, and for some $x_0 \in X$ and all $n \in \mathbb{N}$:

$$d_{X,M}(f^n(x_0), f^{n+1}(x_0)) \leq \lambda_n(d_{X,M}(f^{n-1}(x_0), f^n(x_0))) \quad (1)$$

and $\{(\prod_{i=1}^n \lambda_i)(d_{X,M}(x_0, f(x_0)))\} \in \mathcal{CS}$.

Definition 10. Let M be a partially ordered monoid and $N \subset M_+$. We say that N is bounded, if there exists $m \in M$ such that $n \leq m$ for all $n \in N$.

Definition 11. Let M be a partially ordered monoid and $Z(M)$ be a family of null sequences in M . The pair $(M, Z(M))$ is said to satisfy the Weierstrass property, if for arbitrary sequence $\{\alpha_n\} \subset M$ such that the sequence of partial sums $\{\sum_{k=1}^n \alpha_k\}$ is bounded, one has $\{\alpha_n\} \in \mathcal{CS}$.

Definition 12. We say that a function f is weakly orbitally continuous, if

$$x \in X, f^n(x) \rightarrow a \text{ as } n \rightarrow \infty \implies f(a) = a.$$

The following theorem holds [1].

Theorem 2. Let $(X, d_{X,M}, Z(M))$ be an \mathcal{W} -complete EDM -distance space such that the pair $(M, Z(M))$ satisfies the Weierstrass property, and $f: X \rightarrow X$ be an orbitally continuous operator. Assume that mappings $\phi: X \rightarrow M_+$ and $\eta: M_+ \rightarrow M_+$ are such that for all $x \in X$

$$\eta(d_{X,M}(x, f(x))) + \phi(f(x)) \leq \phi(x),$$

η is semi-additive (i.e. $\eta(a+b) \leq \eta(a) + \eta(b)$ for all $a, b \in M_+$), and for each $A \subset M_+$, if $\eta(A)$ is bounded, then A is bounded. Then the operator f has a fixed point.

[1] Babenko V., Babenko V., Kovalenko O., Fixed points theorems in Hausdorff M -distance spaces, *Fixed Point Theory* **27** (2026), no. 1, 111–132.