ON WEAK SOLUTIONS FOR A SINGULAR BEAM EQUATION

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This result deals with a dynamic Gao beam of infinite length subjected to a moving concentrated Dirac mass. Under appropriate regularity assumptions on the initial data, the problem possesses a weak solution which is obtained as the limit of a sequence of solutions of regularized problems.

The behavior of a beam plays a crucial role in various applications as in railway track design. The railway companies aim to enhance the train rolling speed to meet the increased demands in both passenger and freight transportation worldwide. Identifying the factors contributing to the occurrence of railway track defects is rather important to maintain the required track quality. The oscillation amplitudes in railway tracks due to train movement is intensively studied in scientific literature (see [2, 4]). These oscillations lead to undesirable consequences such as premature wear and deformation of railway tracks. Understanding the impact of these oscillations on system reliability is essential to maintain the track quality necessary for traffic safety and passenger comfort.

We consider a straight infinite Gao beam of thickness $2h$. A horizontal traction p, which is a time-dependent function, is applied at one end. We are focused on examining the sideways movement of a concentrated load $P(t)$, where the load may change over time. This load is applied to a mobile mass m positioned at $\zeta(t)$ with a horizontal velocity $\dot{\zeta}(t)$, induced by a horizontal applied force. The transverse displacement of the Gao beam $u(t, x)$ for $(t, x) \in (0, T) \times \mathbb{R}$ is governed by the following partial differential equation:

$$\begin{aligned} &\varrho u_{tt} + m\delta(x - \zeta(t))u_{tt} - m\delta'(x - \zeta(t))\dot{\zeta}(t)u_t + ku_{xxxx} - (eu_x^2 - \nu p)u_{xx} \\ &= \varrho f + \delta(x - \zeta(t))P(t), \end{aligned} \tag{1}$$

where $\delta(\cdot)$ is the Dirac function, ϱ is the material density, f is the applied mechanical loading, $k = \frac{2h^3E}{3(1-\bar{\nu}^2)}$, $\nu = (1+\bar{\nu})$ and $e = 3hE$ where E and $\bar{\nu}$ denote the Young modulus and the Poisson ratio, respectively. Here and below $(\cdot)_t = \partial_t(\cdot)$, $(\cdot)_x = \partial_x(\cdot)$ denote the partial derivatives with respect to t and x, respectively, while $\dot{(\cdot)}$ and $(\cdot)'$ denote the derivatives with respect to t and x, respectively. We prescribe also initial data

$$u(0, \cdot) = u_0 \quad \text{and} \quad u_t(0, \cdot) = u_1.$$

Equation (1) can be considered the limit of the equation treated in [3] when the gamma viscosity tends to zero, and is related to the usual viscous beam equation without Dirac measure in the time derivative. Nevertheless, this leads to an entirely different mathematical problem since three-order estimates cannot be obtained using the viscous term, moreover, the usual energy estimates fail. In particular, the non-linear term should be addressed somewhat indirectly since no $\L^\infty$ estimate is readily available.

The positive constants ϱ, m, k and e play no role in the mathematical analysis carried out below. Consequently, without loss of generality, we set them equal to 1. Notice, however, that the case of non-constant coefficients (for example $k = k(x)$) could be processed using approximate square roots for elliptic operators.

We assume that $\zeta \in C^2([0,T])$, $P \in C^2([0,T])$, $p \in C^0([0,T];H^2(\mathbb{R}))$ and $f \in C^0([0,T];H^2(\mathbb{R}))$, but these assumptions could be weakened.

Theorem 1. *Let $T>0$. Assume that $u_0 \in H^2(\mathbb{R})$, $u_1 \in H^1(\mathbb{R})$, $\zeta \in C^2([0,T])$, $P \in C^2([0,T])$ and $f,p \in C^0([0,T];H^2(\mathbb{R}))$. Then there exists a function $u \in C^0([0,T);H^{2-\alpha}_{\mathrm{loc}}(\mathbb{R})) \cap L^\infty([0,T];H^2(\mathbb{R}))$, for any $\alpha \in]0,2[$, and $u_t \in L^\infty([0,T];L^2(\mathbb{R}))$ such that for any $v \in C^2([0,T];$
$H^2(\mathbb{R}))$ with a compact support in $[0,T[\times\mathbb{R}$, we have*

$$\begin{cases} \int_0^T \int_{\mathbb{R}} u_t v_t \mathrm{dxdt} + \int_0^{\mathrm{T}} \int_{\mathbb{R}} \mathrm{u_{xx} v_{xx} dxdt} + \frac{1}{3}\int_0^{\mathrm{T}} \int_{\mathbb{R}} \mathrm{u_x^3 v_x dxdt} \\ -\int_0^T \int_{\mathbb{R}} \nu p u_{xx} v \mathrm{dxdt} - \mathrm{u_0(\zeta(0)) v_t(0,\zeta(0))} - \int_0^{\mathrm{T}} \mathrm{u(t,\zeta(t)) v_{tt}(t,\zeta(t))dt} \\ -\int_0^T \dot\zeta(t) u_x(t,\zeta(t)) v_t(t,\zeta(t))\mathrm{dt} - \int_0^{\mathrm{T}} \dot\zeta(\mathrm{t})\mathrm{u(t,\zeta(t)) v_{xt}(t,\zeta(t))dt} \\ +\int_0^T \int_{\mathbb{R}} f v \mathrm{dxdt} + \int_0^{\mathrm{T}} \mathrm{P(t)v(t,\zeta(t))dt} + \int_{\mathbb{R}} \mathrm{u_1 v(0,\cdot)dx} \\ +u_1(\zeta(0))v(0,\zeta(0)) = 0. \end{cases}$$

Moreover, $u(0,\cdot) = u_0$ holds.

In this statement, we refrain from expressing the term $\int_0^T \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{dt}}(\delta(x-\xi(t))\partial_t u(t,\xi(t)))v(t,x)\mathrm{dxdt}$ as $-\int_0^T \partial_t u(t,\xi(t))\partial_t v(t,\xi(t))\mathrm{dt}$. Indeed, the trace is not defined within our functional frame. To establish such an expression, higher-order regularity would be required. However, in the linear homogenous case, the formula $\partial_t^{(2)} u(t,x) + \partial_x^{(4)} u(t,x) = -\frac{\mathrm{d}}{\mathrm{dt}}(\delta(x-\zeta(t))\partial_t u(t,x))$ entails that $\partial_t^{(2)} u$ and $\partial_x^{(4)} u$ cannot both possess $L^2(0,T;L^2(\mathbb{R}))$ regularity. The same formula suggests that $\partial_t^{(2)} u \notin C^0([0,T];L^2(\mathbb{R}))$ as the formal initial data $\partial_t^{(2)} u(0,x) = -(\partial_x^{(4)} u(0,x) - \frac{\mathrm{d}}{\mathrm{dt}}(\delta(x-\zeta(0))\partial_t u(0,x)))$ does not belong to $L^2(\mathbb{R})$ for a smooth data $u_0(x) = u(0,x)$. Lastly, the usual energy estimate, obtained by multiplying (1) by $\partial_t u$ and by integrating the result over $[0,T]\times\mathbb{R}$, can not be performed due to the term $\delta(x-\zeta(t))$ [1].

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[1] Atlasiuk O., Heibig A., Petrov A., Weak solutions for a singular beam equation, *Zeitschrift für Angewandte Mathematik und Mechanik* **105** (2025), 12, 18 pp, https://doi.org/10.1002/zamm.70296.

[2] Dimitrovová Z., Rodrigues A., Critical velocity of a uniformly moving load, *Adv. Eng. Softw.* **50** (2012), 44–56.

[3] Dyniewicz B., Bajer C.I.,Kuttler K.L., Shillor M., Vibrations of a Gao beam subjected to a moving mass, *Nonlinear Analysis: Real World Appl.* **50** (2019), 342–364.

[4] Frýba L. Vibration of Solids and Structures under Moving Loads. Prague: Research Institute of Transport (1972).