Joint asymptotic normality of drift estimators for the CKLS Model

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We consider the stochastic differential equation

$$dr_t = (a - br_t)\,dt + \sigma r_t^{\beta}\,dW_t, \tag{1}$$

where $W = \{W_t, t \geq 0\}$ is a Wiener process and the parameters satisfy

$$a, b, \sigma > 0, \qquad \beta \in (1/2, 1), \qquad r_0 > 0. \tag{2}$$

Equation (1) is known as the Chan–Karolyi–Longstaff–Sanders (CKLS) model, introduced in [1] for interest rate modeling. The limiting case $\beta = \frac{1}{2}$ corresponds to the classical Cox–Ingersoll–Ross process. The CKLS model provides a flexible class of diffusion models in which the volatility coefficient depends on the current level of the process through a power function. This feature is important in financial applications, since it allows one to model different degrees of dependence of volatility on the state variable, ranging from square-root type behavior to nearly linear diffusion coefficients. From the statistical point of view, however, the nonlinear diffusion structure makes the analysis of drift estimators more delicate than in the classical CIR case.

We study the estimation of the unknown drift parameters (a, b) from continuous observations of a trajectory $\{r_t, t \in [0, T]\}$. The diffusion parameters σ and β are assumed to be known. This assumption is natural in the continuous-time observation setting, since σ and β can be evaluated almost surely, without prior knowledge of a and b, as discussed in detail in [3].

In this work, we investigate the following strongly consistent drift estimators introduced in [3]:

$$\widetilde{a}_T = \frac{\sigma^2(1-\beta)\left(\int_0^T r_t\,dt\right)^2}{T\int_0^T r_t^{3-2\beta}\,dt - \left(\int_0^T r_t\,dt\right)\left(\int_0^T r_t^{2-2\beta}\,dt\right)},$$
$$\widetilde{b}_T = \frac{\sigma^2(1-\beta)T\int_0^T r_t\,dt}{T\int_0^T r_t^{3-2\beta}\,dt - \left(\int_0^T r_t\,dt\right)\left(\int_0^T r_t^{2-2\beta}\,dt\right)}.$$

Our main objective is to prove the joint asymptotic normality of these estimators.

Under assumptions (2), the solution $r = \{r_t, t \geq 0\}$ is strictly positive and ergodic. Its invariant density is given by

$$p_\infty(x) = Gx^{-2\beta}\exp\left\{\frac{2}{\sigma^2}\left(\frac{ax^{1-2\beta}}{1-2\beta} - \frac{bx^{2-2\beta}}{2-2\beta}\right)\right\}, \qquad x > 0,$$

where G is the normalizing constant. Moreover, all invariant power moments are finite:

$$\mu_\alpha := \int_0^\infty x^\alpha p_\infty(x)\,dx < \infty, \qquad \alpha \in \mathbb{R}.$$

Therefore, by the ergodic theorem,

$$\frac{1}{T}\int_0^T r_t^\alpha\,dt \to \mu_\alpha \qquad \text{a.s. as } T \to \infty. \tag{3}$$

The following theorem presents our main result.

Theorem 1. *As $T \to \infty$,*

$$\sqrt{T}\begin{pmatrix}\widetilde{a}_T - a\\ \widetilde{b}_T - b\end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$

where

$$\Sigma = \frac{b^4}{\sigma^2(1-\beta)^2 a^2}\, A\Gamma A^\top,$$

with

$$A = \begin{pmatrix}\mu_{3-2\beta} & a/b\\ \mu_{2-2\beta} & 1\end{pmatrix}, \qquad \Gamma = \begin{pmatrix}\mu_{2\beta} & -\mu_2\\ -\mu_2 & \mu_{4-2\beta}\end{pmatrix}.$$

The obtained result complements the strong consistency theorem from [3] by providing the next-order asymptotic behavior of the estimators. In particular, it identifies the asymptotic covariance matrix explicitly in terms of invariant moments of the CKLS process, thereby showing how the diffusion parameters and the long-term distribution of the model affect the asymptotic accuracy of drift estimation.

The proof is based on a representation of the normalized estimation error in terms of the two-dimensional Itô martingale

$$\left(T^{-1/2}\int_0^T r_t^{\beta}\, dW_t,\ -T^{-1/2}\int_0^T r_t^{2-\beta}\, dW_t\right)^\top.$$

The joint asymptotic normality of this martingale follows from the multidimensional martingale central limit theorem, while the convergence of its quadratic variation is obtained from the ergodic relation (3).

As an auxiliary result of independent interest, we prove the uniform boundedness of all positive moments of the process r.

Lemma 1. *Let assumptions* (2) *hold. Then for any $p > 0$ there exists a constant $C_p > 0$ such that*

$$\sup_{t\ge 0} \mathbb{E}\, r_t^p \le C_p.$$

This lemma extends [2, Lemma 3.8], where the corresponding bound was established for $p = 2$.

[1] Chan K.C., Karolyi G.A., Longstaff F.A., Sanders A.B., An empirical comparison of alternative models of the short-term interest rate, *J. Finance* **47** (1992), 1209–1227.

[2] Mishura Y., Pilipenko A., Ralchenko K., Gatheral double stochastic volatility model with Skorokhod reflection, *Theory Probab. Math. Statist.* **113** (2025), 153–171.

[3] Mishura Y., Ralchenko K., Dehtiar O., Parameter estimation in CKLS model by continuous observations, *Statist. Probab. Lett.* **184** (2022), Paper No. 109391.