Group classification of a class of systems of nonlinear diffusion equations

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We investigate a class of $(1+1)$-dimensional systems of nonlinear diffusion equations from the viewpoint of Lie symmetry analysis. Our main focus is on constructing the equivalence group of the class, identifying the kernel of the maximal Lie invariance algebras, singling out nine inequivalent subclasses, whose systems admit Lie symmetry extensions, and performing the complete group classification for one of these subclasses.

The class consists of systems of diffusion equations of the form

$$U_t = \partial_x\left[F(U)U_x\right], \tag{1}$$

where $U = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in \mathbb{R}^2$ and the matrix $F(U) = \begin{pmatrix} f^{11} & f^{12} \\ f^{21} & f^{22} \end{pmatrix}$, $f^{ab} = f^{ab}(u^1, u^2)$ are the diffusion coefficients, $u^a = u^a(t,x)$ are the concentration (or density) functions of two different substances that interact with each other, t and x are time and spatial independent variables, respectively.

The kernel of the maximal Lie invariance algebras of the system (1) is the three-dimensional algebra

$$A^{\cap} = \langle \partial_t,\, \partial_x,\, D = 2t\partial_t + x\partial_x \rangle. \tag{2}$$

Theorem 1. *The equivalence algebra of the class (1) is the linear span of the vector fields*

$$\partial_t,\ \partial_x,\ \partial_{u^a},\ t\partial_t - f^{ab}\partial_{f^{ab}},\ x\partial_x + 2f^{ab}\partial_{f^{ab}},\ u^a\partial_{u^b} + f^{ac}\partial_{f^{bc}} - f^{cb}\partial_{f^{ca}}$$

in the space $\mathbb{R}^2_{t,x} \times \mathbb{R}^2_U \times \mathbb{R}^4_F$, which is isomorphic to $\mathfrak{aff}(1,\mathbb{R}) \oplus \mathfrak{aff}(1,\mathbb{R}) \oplus \mathfrak{aff}(2,\mathbb{R})$.

Corollary 1. *The maximal group of continuous equivalence transformations of the class (1) consists of point transformations.*

$$\tilde{t} = \mathrm{e}^{\theta_1}t + s_0, \quad \tilde{x} = \mathrm{e}^{\theta_2}x + s_1, \quad \tilde{U} = AU + Q, \quad \tilde{F} = \mathrm{e}^{\theta_1 - 2\theta_2}A^{-1}FA \tag{3}$$

in the space $\mathbb{R}^2_{t,x} \times \mathbb{R}^2_U \times \mathbb{R}^4_F$, where $A = \begin{pmatrix} \mathrm{e}^{\theta_3} + \theta_4\theta_5\mathrm{e}^{-\theta_6} & \theta_4 \\ \theta_5 & \mathrm{e}^{\theta_6} \end{pmatrix}, Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, and s_0, s_1, q_1, q_2, θ_0, $\theta_1, \ldots, \theta_6$ are arbitrary constants.

Remark 1. In addition to the continuous transformations (3), the class of systems of equations (1) also admits discrete equivalence transformations of the form

$$\begin{aligned}
&\tilde{t} = -t,\ \tilde{x} = x,\ \tilde{u}^a = u^a,\ \tilde{f}^{ab} = -f^{ab},\\
&\tilde{t} = t,\ \tilde{x} = -x,\ \tilde{u}^a = u^a,\ \tilde{f}^{ab} = f^{ab},\\
&\tilde{t} = t,\ \tilde{x} = x,\ \tilde{u}^1 = -u^1,\ \tilde{u}^2 = u^2,\ \tilde{f}^{11} = f^{11},\ \tilde{f}^{12} = -f^{12},\ \tilde{f}^{21} = -f^{21},\ \tilde{f}^{22} = f^{22},\\
&\tilde{t} = t,\ \tilde{x} = x,\ \tilde{u}^1 = u^1,\ \tilde{u}^2 = -u^2,\ \tilde{f}^{11} = f^{11},\ \tilde{f}^{12} = -f^{12},\ \tilde{f}^{21} = -f^{21},\ \tilde{f}^{22} = f^{22}.
\end{aligned} \tag{4}$$

The superposition of transformations (3) and (4) defines transformations

$$\tilde{t} = l_0t + s_0, \quad \tilde{x} = l_1x + s_1, \quad \tilde{U} = PU + Q, \quad \tilde{F} = \frac{l_0}{l_1^2}P^{-1}FP \tag{5}$$

in the space $\mathbb{R}^2_{t,x} \times \mathbb{R}^2_U \times \mathbb{R}^4_F$, where $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, and l_0, l_1, p_{ab} are arbitrary constants such that $l_0 l_1 \neq 0$, P is an arbitrary constant nondegenerate matrix.

Theorem 2. *Systems of equations of the class (1) admit extensions of the kernel of maximal Lie invariance algebras (MIA) (2) if and only if the nonlinearities $f^{ab} = f^{ab}(u^1, u^2)$, up to the equivalence induced by transformations (5), take the forms listed in Table 1.*

Table 1: The types of nonlinearities for which systems (1) admit the extension of the kernel (2) and the kernels of MIA for the respective subclasses.

No	f^{ab}	The kernel of MIA
1	$f^{ab} = \mathrm{e}^{nu^1}\varphi^{ab},\ \ \omega = u^2$	$A^\cap_1 = \langle A^\cap, nt\partial_t - \partial_{u^1}\rangle,\ \ n \in \{0,1\}$
2	$f^{11} = \mathrm{e}^{n\frac{u^1}{u^2}}\left(\varphi^{11} + \frac{u^1}{u^2}\varphi^{21}\right)$, $f^{12} = \mathrm{e}^{n\frac{u^1}{u^2}}\left(\varphi^{12} + \frac{u^1}{u^2}\left(\varphi^{22} - \varphi^{11}\right) - \left(\frac{u^1}{u^2}\right)^2\varphi^{21}\right)$, $f^{21} = \mathrm{e}^{n\frac{u^1}{u^2}}\varphi^{21},\ \ f^{22} = \mathrm{e}^{n\frac{u^1}{u^2}}\left(\varphi^{22} - \frac{u^1}{u^2}\varphi^{21}\right),\ \ \omega = u^2$	$A^\cap_2 = \langle A^\cap, nt\partial_t - u^2\partial_{u^1}\rangle$, $n \in \{0,1\}$
3	$f^{11} = (u^1)^n\varphi^{11},\ \ f^{12} = (u^1)^{n+1}\varphi^{12}$, $f^{21} = (u^1)^{n-1}\varphi^{21},\ \ f^{22} = (u^1)^n\varphi^{22},\ \ \omega = u^2$	$A^\cap_3 = \langle A^\cap, nt\partial_t - u^1\partial_{u^1}\rangle,\ \ n \in \mathbb{R}$
4	$f^{11} = (u^2)^n\,\varphi^{11},\ \ f^{12} = (u^2)^{n-1}\,\varphi^{12}$, $f^{21} = (u^2)^{n+1}\,\varphi^{21},\ \ f^{22} = (u^2)^n\,\varphi^{22},\ \ \omega = u^1 - \ln u^2$	$A^\cap_4 = \left\langle A^\cap, nt\partial_t - u^2\partial_{u^2} - \partial_{u^1}\right\rangle$, $n \in \mathbb{R}$
5	$f^{11} = \mathrm{e}^{nu^1}(\varphi^{11} - u^1\varphi^{12}),\quad f^{12} = \mathrm{e}^{nu^1}\varphi^{12}$, $f^{21} = \mathrm{e}^{nu^1}\left(\varphi^{21} + u^1(\varphi^{11} - \varphi^{22}) + (u^1)^2\varphi^{12}\right)$, $f^{22} = \mathrm{e}^{nu^1}(\varphi^{22} + u^1\varphi^{12}),\ \ \omega = u^2 + 1/2(u^1)^2$	$A^\cap_5 = \left\langle A^\cap, nt\partial_t - u^1\partial_{u^2} - \partial_{u^1}\right\rangle$, $n \in \{0,1\}$
6	$f^{11} = (u^1)^n\varphi^{11},\ \ f^{12} = (u^1)^{n-m}\varphi^{12}$, $f^{21} = (u^1)^{n+m}\varphi^{21},\ \ f^{22} = (u^1)^n\varphi^{22},\ \ \omega = \frac{u^2}{(u^1)^{m+1}}$	$A^\cap_6 = \langle A^\cap, nt\partial_t - I - mu^2\partial_{u^2}\rangle$, $n \in \mathbb{R},\quad m \in \mathbb{R},\quad m \neq -1, 0$
7	$f^{ab} = (u^1)^n\varphi^{ab},\ \ \omega = u^2/u^1$	$A^\cap_7 = \langle A^\cap nt, \partial_t - I\rangle,\ \ n \in \mathbb{R}$
8	$f^{11} = (u^2)^n\,(\varphi^{11} + \varphi^{21}\ln u^2)$, $f^{12} = (u^2)^n\,\left(\varphi^{12} + (\varphi^{22} - \varphi^{11})\ln u^2 - \varphi^{21}\ln^2 u^2\right)$, $f^{21} = (u^2)^n\,\varphi^{21}$, $f^{22} = (u^2)^n\,(\varphi^{22} - \varphi^{21}\ln u^2),\ \ \omega = u^1/u^2 + \ln u^2$	$A^\cap_8 = \langle A^\cap, nt\partial_t - I - u^2\partial_{u^1}\rangle$, $n \in \mathbb{R}$
9	$f^{1a} = \mathrm{e}^{-n\theta}\left(2\rho^{-2}u^a\vec{u}\vec{\varphi} + \psi^a\right)$, $f^{2a} = \mathrm{e}^{-n\theta}\left(2\rho^{-2}u^a\vec{u}\vec{\varphi}^\perp + \psi^{a^\perp}\right),\ \ \omega = \rho\mathrm{e}^{m\theta}$	$A^\cap_9 = \langle A^\cap, nt\partial_t - mI - J\rangle$, $n \in \mathbb{R},\quad m \in \mathbb{R}$

Throughout Table 1 the functions $\varphi^{ab} = \varphi^{ab}(\omega)$ are arbitrary smooth functions satisfying the conditions $\varphi^{11} + \varphi^{22} \neq 0$, $\varphi^{11}\varphi^{22} - \varphi^{12}\varphi^{21} \neq 0$, $I = u^a\partial_{u^a}$, $J = u^2\partial_{u^1} - u^1\partial_{u^2}$. In entry 9 of Table 1 the functions $\varphi^a = \varphi^a(\omega)$, $\psi^a = \psi^a(\omega)$ are arbitrary smooth functions such that $\varphi^1 \neq -\psi^1$, $\vec{\psi}^2 + 2\vec{\varphi}\vec{\psi} \neq 0$, $\rho = \sqrt{u^a u^a}$, $\theta = \arctan u^2/u^1$, $\vec{u} = (u^1, u^2)$, $\vec{\varphi} = (\varphi^1, \varphi^2)$, $\vec{\psi} = (\psi^1, \psi^2)$, $\vec{\varphi}^\perp = (-\varphi^2, \varphi^1)$, $\vec{\psi}^\perp = (-\psi^2, \psi^1)$.

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