Approximation of classes of periodic functions of several variables with given majorant of mixed moduli of continuity in the space L_∞

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The purpose of the work is to obtain the estimates of one approximation characteristic of the Nikol'skii-Besov-type classes $B^{\Omega}_{\infty,\theta}$ of periodic functions of several variables with a given function $\Omega(t)$ of a special form. Approximation of functions from corresponding classes is carried out using linear operators that are subject to certain conditions, and at the same time the approximation error is estimated in the space L_∞. The obtained estimates complement and generalize some of the results of the work [1], related to approximation of the Nikol'skii-type classes H^{Ω}_{∞}.

Let $\mathbb{R}^d, d \geq 1$, be a d-dimensional space with elements $x = (x_1, \ldots, x_d)$ and $(x, y) = x_1y_1 + \ldots + x_dy_d$ be the scalar product of the elements $x, y \in \mathbb{R}^d$. Let $L_p(\mathbb{T}^d)$, $\mathbb{T}^d = \prod\limits_{j=1}^{d} [0, 2\pi)$, denote the space of functions f that are 2π-periodic in each variable and for which

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} = \left((2\pi)^{-d} \int\limits_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} = \operatorname*{ess\,sup}_{x \in \mathbb{T}^d} |f(x)| < \infty.$$

Let $1 \leq p \leq \infty,\ 1 \leq \theta \leq \infty$, and let $\Omega(t)$ be a given function of the type of a mixed modulus of continuity of the order l. Then the classes $B^{\Omega}_{p,\theta}$ are defined in the following way [2]:

$$B^{\Omega}_{p,\theta} = \left\{ f \in L^0_p(\mathbb{T}^d) : \ \|f\|_{B^{\Omega}_{p,\theta}} \leq 1 \right\},$$

where

$$\|f\|_{B^{\Omega}_{p,\theta}} = \left\{ \int\limits_{\mathbb{T}^d} \left(\frac{\Omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \prod_{j=1}^{d} \frac{dt_j}{t_j} \right\}^{1/\theta}, \quad 1 \leq \theta < \infty,$$

$$\|f\|_{B^{\Omega}_{p,\infty}} = \sup_{t>0} \frac{\Omega_l(f,t)_p}{\Omega(t)},$$

(the expression $t > 0$ for $t = (t_1, ..., t_d)$ is equivalent to $t_j > 0, j = \overline{1,d}$). In what follows, we study the classes $B^{\Omega}_{p,\theta}$ that are defined by the function $\Omega(t)$:

$$\Omega(t) = \Omega(t_1, ..., t_d) = \begin{cases} \prod\limits_{j=1}^{d} \dfrac{t_j^r}{(\log 1/t_j)_+^{b_j}}, & \text{if } t_j > 0,\ j = \overline{1,d}; \\ 0, & \text{if } \prod\limits_{j=1}^{d} t_j = 0. \end{cases} \tag{1}$$

Here and below, we consider the logarithms with base 2, and

$$(\log(1/t_j))_+ = \max\{1, \log(1/t_j)\}.$$

For the functional class $F \subset L_q(\mathbb{T}^d)$ we denote :

$$d_M^B(F, L_q) = \inf_{G \in L_M(B)_q} \sup_{f \in F \cap D(G)} \|f - Gf\|_q,$$

where $L_M(B)_q$ is the set of linear operators satisfying the following conditions:

a) the domain of definition $D(G)$ of these operators contains all trigonometric polynomials, and their domain of values is contained in a subspace with dimension M of the space $L_q(\mathbb{T}^d)$;

b) there exists a number $B \geq 1$ such that, for all vectors $k = (k_1, \ldots, k_d)$, $k_j \in \mathbb{Z}$, $j = \overline{1, d}$, the inequality $\left\|Ge^{i(k,\cdot)}\right\|_2 \leq B$ holds.

Theorem 1. *Let $1 \leq \theta < \infty$ and $\Omega(t)$ be a function of the form (1). Then, for $0 < r < l$, $b_1 \leq \ldots \leq b_d < r$, the relation*

$$d_M^B(B_{\infty,\theta}^\Omega, L_\infty) \asymp M^{-r} (\log M)^{-b_1 - \ldots - b_d + (d-1)(r+1-1/\theta)}$$

holds.

Theorem 2. *Let $1 \leq \theta < \infty$ and $\Omega(t)$ be a function of the form (1). Then, for $0 < r < l$, $b_j > r + 1,\ j = \nu + 1, \ldots, d$, the relation*

$$d_M^B(B_{\infty,\theta}^\Omega, L_\infty) \asymp M^{-r} \big(\log M\big)^{-b_1 - \ldots - b_\nu + (\nu-1)(r+1-1/\theta)}$$

holds.

Theorem 3. *Let $1 \leq \theta < \infty$ and $\Omega(t)$ be a function of the form (1). Then, for $0 < r < l$, $b_2 > b_1 + 1$, the following relation holds:*

$$d_M^B(B_{\infty,\theta}^\Omega, L_\infty) \asymp M^{-r} \big(\log M\big)^{-b_1}.$$

Remark 1. The analogues of Theorems 1–3 for the classes H_∞^Ω are obtained in [1]. Moreover, if the conditions of Theorem 3 are fulfilled, the following relation hold:

$$d_M^B(B_{\infty,\theta}^\Omega, L_\infty) \asymp d_M^B(H_\infty^\Omega, L_\infty).$$

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2. Yongsheng S., Heping W. Representation and approximation of multivariate periodic functions with bounded mixed moduli of smoothness. Tr. Mat. Inst. Steklova 1997, **219**, 356–377.