PERTURBATION ANALYSIS OF SINGULAR VALUES IN CONCATENATED MATRICES

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Concatenating matrices is a powerful strategy for revealing common structures in data. For instance, stacking k matrices $A_1, \dots, A_k$ into a single block matrix

$$M = [A_1\ A_2\ \cdots\ A_k],$$

allows us to extract shared basis vectors through a single singular value decomposition (SVD). This approach underlies many algorithms in dimensionality reduction, clustering, and other data-driven applications [3, 4].

However, an important question is how perturbations in the individual matrices A_i affect the singular values of the concatenated matrix M. Building on classical results in matrix perturbation theory [1, 2], we derive explicit error bounds that quantify this sensitivity.

Theorem 1. *Let $M = [A_1, \dots, A_k]$ be formed by horizontally stacking matrices $A_i \in \mathbb{R}^{m\times n}$. Suppose $\widetilde{M} = [\widetilde{A}_1, \dots, \widetilde{A}_k]$ is a perturbed version, where $\widetilde{A}_i = A_i + E_i$. Denote $\sigma_i(M)$ and $\sigma_i(\widetilde{M})$ their respective singular values, and let $r = \operatorname{rank}(M)$. Then for $1 \le i \le r$,*

$$|\sigma_i(\widetilde{M}) - \sigma_i(M)| \le \frac{1}{\sigma_i(M)} \sum_{j=1}^{k} \bigl(2\,\|A_j\|_2\,\|E_j\|_2 + \|E_j\|_2^2\bigr),$$

and for $i > r$,

$$\sigma_i(\widetilde{M}) \le \sqrt{\sum_{j=1}^{k} \bigl(2\,\|A_j\|_2\,\|E_j\|_2 + \|E_j\|_2^2\bigr)}.$$

Remark 1. *These bounds ensure that small blockwise perturbations E_i lead to only modest changes in the dominant singular values of M. Hence, if $\|E_i\|_2$ is sufficiently small for each i, $\sigma_i(\widetilde{M})$ remains close to $\sigma_i(M)$ for all indices, preserving the reliability of subsequent low-rank approximations and clustering tasks.*

Practical Impact. Concatenation-based approaches appear in a broad range of problems, from multisensor signal processing to large-scale matrix clustering. By focusing on explicit spectral norm estimates, our results allow practitioners to decide when joint compression (via concatenation) is beneficial, or when separate compression suffices.

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